Centro de Investigación en Matemáticas, A.C.

HANAMURA'S COHOMOLOGICAL CHOW GROUPS AND A REGULATOR MORPHISM TO ABSOLUTE HODGE COHOMOLOGY

T E S I S

Que para obtener el grado de

Doctor en Ciencias

con Orientación en

Matemáticas Básicas

Presenta

Diosel López Cruz

Directores de Tesis:

Dr. Pedro Luis Del Ángel Rodríguez

Dr. Xavier Gómez-Mont Ávalos

Autorización de la versión final

Guanajuato, Gto., 05 de Febrero de 2021

Abstract

The purpose of this work is to describe some fundamental properties of Hanamura's cohomological Chow groups. In the singular case, we establish an analogue of Voevodsky's comparison theorem between Hanamura's motivic cohomology and Friedlander–Voevodsky's motivic cohomology. Using this identification and the Kerr–Lewis–Müller-Stach construction (KLM-formula), we construct a regulator for singular (quasi-projective) varieties, from motivic cohomology to absolute Hodge cohomology. The procedure is via cubical hyperresolutions $X_\bullet \to X$ of Guillén–Navarro–Pascual–Puerta.

In the end, we do some explicit calculations of motivic cohomology, using Hanamura's spectral sequence. If we consider varieties of dimension three with smooth singular locus, this reduces the calculations to the normal crossing divisor. Then we consider varieties of higher dimensions, with the same conditions as for the previous varieties.

Alguien dijo:
-¡...ámonos!
Los tres novilleros echaron a caminar, al frente de sus cuadrillas. A mitad del ruedo Luis Ortega se descubrió, por ser el debutante.
En ese momento, dominando la ovación y el pasodoble, el tiempo golpeó cuatro veces en la campana del reloj de la Plaza México.
¡¡ La hora !!
Y Luis Ortega ponía ya el primer paso en el misterio.

Luis Spota, "Más cornadas da el hambre" (1950)

Acknowledgements

Quiero empezar diciendo que este trabajo no es solo mío, en él han intervenido directa e indirectamente muchas personas. Agradezco en primer lugar a mis padres, Estela y Miguel, por todo el apoyo que me han brindado durante toda mi carrera como matemático, sin ellos esto no sería posible. A mi hermanos que también han sido un pilar fundamental en mi formación, como persona y como matemático. Gracias por quererme tanto.

Reconozco también a mis asesores, Pedro Luis del Ángel y Xavier Gómez-Mont, por haberme guiado por este apasionante camino de las matemáticas. Por dotarme de esas botas para llegar a la cima de la montaña. Agradezco también a todos mis sinodales, Manuel Gónzalez Villa, Jaime Hernández, Alexey Beshenov, James D. Lewis y Pablo Pelaez. Gracias por tomarse el tiempo para leer este trabajo. Muy en especial, agradezco a Manuel Gónzalez, Jaime Hernández y Alexey Beshenov, por ayudarme con cualquier duda que surgió durante este trabajo. Gracias por estar ahí para cualquier inquietud acerca de teoría de Hodge, ciclos algebraicos, cohomología motívica, y con cualquier problema tipográfico relacionado con la escritura de la tesis.

Mención especial merecen todos los amigos que estuvieron durante esta etapa de mi vida, los que hice en el Centro de Investigación en Matemáticas (CIMAT) y en la ciudad de Guanajuato. A esa familia que formamos con el equipo de fútbol Peyote, por todas las alegrias y campeonatos que complementaban perfectamente mi labor académica con un poco de esparcimiento. Finalmente, quiero agradecer al Consejo Nacional de Ciencia y Tecnología (CONACYT) por el apoyo brindado durante mi estancia como estudiante de doctorado. Asimismo, quisiera agradecer al CIMAT por el apoyo brindado para la culminación de este trabajo. Gracias a todos...

Contents

Table of notations

$H^l_{\mathcal{H}}(K^\bullet)$	absolute Hodge cohomology of $\mathbb{A}$-MHC $K^\bullet$
$\mathcal{Z}^r(X)$	algebraic cycles
$\mathcal{Z}^r(X,\bullet)$	Bloch's cycle complex
$\mathrm{CH}^r(X,m)$	Bloch's higher Chow groups
$H^{BM}_p(X,\mathbb{Z}(q))$	Borel-Moore motivic homology
$\mathbf{Corr}_{\mathrm{fin}}(k)$	category of finite correspondences
$\mathrm{DM}_{\mathrm{gm}}(k)$	category of geometric motives
MHS	category of mixed Hodge structures
HS	category of (pure) Hodge structures
$\mathbf{QuProj}(k)$	category of quasi-projective varieties
$\mathbf{SmProj}(k)$	category of smooth, projective varieties
$\mathbf{Sm}(k)$	category of smooth varieties
$\mathrm{CH}^r(X)$	Chow groups
$\mathrm{CHC}^r(X,m)$	Chow cohomology groups
$X_\bullet$	cubical hyperresolution of X
$\mathcal{D}^{p,q}_X(U)$	currents on U
$H^q_{\mathcal{D}}(X,\mathbb{A}(p))$	Deligne cohomology
$\mathbb{A}^\bullet_{\mathcal{D}}(p)$	Deligne complex
$H^*_{\mathcal{DB}}(X,\mathbb{A}(p))$	Deligne-Beilinson cohomology
$\mathbb{A}^\bullet_{\mathcal{DB}}(p)$	Deligne-Beilinson complex
$\mathbf{D}^b(\mathcal{A})$	bounded derived category of $\mathcal{A}$
$\mathbf{D}(\mathbb{Q})$	derived category of $\mathbb{Q}$-vectos spaces
$\mathrm{Jac}^q(X)$	Griffiths' intermediate Jacobian
$\Omega^\bullet_X$	holomorphic differential forms on X
$\mathbf{K}^b(\mathcal{A})$	homotopy category of complexes in $\mathcal{A}$
$H^q_{\mathcal{M}}(X,\mathbb{Z}(p))$	motivic cohomology group
$\mathrm{Pic}(X)$	Picard group of X
$\mathcal{Z}^p_{\mathbb{R}}(X,\bullet)$	real cycles
$\mathbb{Z}^{SF}(i)$	Suslin-Friedlander motivic complex

Introduction

The **category of mixed motives** $\mathrm{MM}(k)$ is conjecturally described by A. Beilinson [Bei87] and P. Deligne, as a tensor abelian category which contains to Grothendieck's category of pure motives as the full subcategory of semi-simple objects, in analogy with the mixed Hodge structures [HodgeI], [HodgeII] and [HodgeIII]. One expected important property is the existence of a Bloch-Ogus cohomology theory for smooth schemes:

$$H^p_M\big(X, \mathbb{Z}(q)\big) := \mathrm{Ext}^p_{\mathrm{MM}(k)}\big(\mathbb{Z}(0), M(X) \otimes \mathbb{Z}(q)\big)$$

with Chern classes from algebraic K-theory:

$$c^{q,p}\colon K_{2q-p}(X) \to H^p_M\big(X, \mathbb{Z}(q)\big)$$

which induce an isomorphism:

$$K_{2q-p}(X)^{(q)} \cong H^p_M\big(X, \mathbb{Z}(q)\big) \otimes \mathbb{Q},$$

where $K_*(-)^{(q)}$ is the weight eigenspace of the Adams operations.

The original idea of **motivic cohomology** as a universal cohomology theory for algebraic varieties is due to A. Grothendieck. This should be a theory that plays the same role in algebraic geometry as singular cohomology in algebraic topology. A. Beilinson [Bei87] and S. Lichtenbaum [Lic84] conjectured, independently, that integral motivic cohomology can be computed as the hypercohomology of adequate complexes $\mathbb{Z}(q)$ on Zariski and étale site on X in the derived category, respectively. As in algebraic topology, Beilinson suggests the existence of an "Atiyah-Hirzebruch spectral sequence" for smooth schemes of the form

$$E_2^{p,q} = \mathbb{H}^{p-q}_{\mathrm{Zar}}(X, \mathbb{Z}(-q)) \Rightarrow K_{-p-q}(X)$$

converging to Quillen's algebraic K-theory whose E_2-term is motivic cohomology. For the properties we refer to [Bei87] and [Lic84].

0.1 Absolute motivic cohomology. The first approximation to motivic cohomology is via K-theory. Guided by the relationship between K-theory and singular cohomology in the topological case, Beilinson [Bei84] defines, for a quasi-projective variety X over $\mathbb{C}$ its rational motivic cohomology as

$$H^q_{\mathrm{mot}}(X, \mathbb{Q}(n)) := K_{2n-q}(X)^{(n)} \otimes \mathbb{Q} \tag{1}$$

the subspace of weight n for Adams operations of a situable K-group of X. Moreover, for a regular complex projective variety X, Beilinson constructed a regulator morphism to the Deligne cohomology of X:

$$\mathrm{reg}_B \colon H^q_{\mathrm{mot}}(X, \mathbb{Q}(n)) \to H^q_{\mathcal{D}}(X, \mathbb{Z}(n)) \to H^q_{\mathcal{D}}(X, \mathbb{R}(n));$$

with applications to special values of L-functions of Grothendieck's motives.

0.2 Higher Chow groups. In the spirit of singular cohomology and in terms of algebraic cycles, S. Bloch [Blo86a] has another approach to motivic cohomology. Bloch's cycle complexes are complexes of sheaves in the Zariski topology. Let $\mathcal{Z}^r(X, m)$ be the subgroup of $\mathcal{Z}^r(X \times \Delta^m)$ generated by the cycles meeting all the faces properly, where $\Delta^m = \mathrm{Spec}\big\{k[t_0, \ldots, t_m]/(1 - \sum_{j=0}^m t_j)\big\}$ is the *algebraic simplex*. The assignment $m \mapsto \mathcal{Z}^r(X, m)$ defines a simplicial abelian group $\mathcal{Z}^r_\Delta(X, \bullet)$. The *higher Chow groups* are defined as

$$\mathrm{CH}^r(X, m) := H_m(\mathcal{Z}^r_\Delta(X, \bullet)).$$

These groups generalize the classical Chow groups $\mathrm{CH}^r(X, 0) = \mathrm{CH}^r(X)$, cycles of codimension r on X modulo rational equivalence. Tensoring with $\mathbb{Q}$, Bloch proves the following identification:

$$\mathrm{CH}^r(X, m) \otimes \mathbb{Q} \cong K_m(X)^{(r)}_{\mathbb{Q}},$$

where $K_m(X)^{(r)}_{\mathbb{Q}}$ is the subspace of weight r (for Adams operations) in the rational Quillen K-theory of X. This generalizes the classical result of Grothendieck identifying $\mathrm{CH}^r(X) \otimes \mathbb{Q} \cong K_0(X)^{(r)}_{\mathbb{Q}}$. Furthermore, it recovers the original definition of Beilinson (1). Then, higher Chow groups are also natural candidates for motivic cohomology. There are several recent approaches to motivic cohomology, many of them in terms of algebraic cycles, among which Friedlander-Voevodsky [FV00], Hanamura [Han00], and Voevodsky [MVW06]. In this direction, Voevodsky proved a fundamental result: Bloch's higher Chow groups are isomorphic to motivic cohomology in the smooth case, for a field k that admits resolution of singularities.

Theorem 0.0.1 [MVW06, Theorem 19.1] *Let $X \in \mathbf{Sm}(k)$. Then motivic cohomology coincides with higher Chow groups*

$$\mathrm{CH}^r(X,m) \cong H^{2r-m}_{\mathcal{M}}(X,\mathbb{Z}(r)) = \mathrm{Hom}_{\mathrm{DM_{gm}(k)}}(M(X),\mathbb{Z}(r)[2r-m]),$$

where $\mathrm{DM_{gm}}(k)$ is the Voevodsky's triangulated category of mixed motives. In particular, for $m = 0$, we recover the fact that $\mathrm{CH}^r(X) = H^{2r}_{\mathcal{M}}(X,\mathbb{Z}(r))$.

0.3 Regulators. Motivic cohomology is related to other theories of absolute cohomologies via *regulators*. The term "regulator" comes from the relationship that these morphisms have with the Borel and Beilinson-Bloch regulators [Bei84]. The regulator of a number field F is a morphism

$$R_F\colon \mathcal{O}^*_F = K_1(\mathcal{O}_F) \to \mathbb{R}^{r_1+r_2-1}$$

used by Dirichlet in his study of the units of the ring of integers of a number field. The *class number formula* is a result proved by Dirichlet that relates all the important numerical invariants of the number field to the covolume of the regulator R_F. Borel (Bloch-Beilinson) regulator is the higher dimensional analogue of the Dirichlet regulator, considered as a morphism from algebraic K-theory to some theory of absolute cohomology.

0.4 Absolute Hodge cohomology. Another main ingredient in the construction of the regulator morphism (and Beilinson's conjectures) is the *absolute Hodge cohomology*. This is defined in terms of a mixed Hodge complex [Bei86] and [HodgeIII]. In fact, the constructions of Deligne cohomology and homology are also given in terms of mixed Hodge complexes. An $\mathbb{A}$-mixed Hodge complex $K^\bullet_{\mathcal{H}}$ is given by a diagram in the derived category:

$$K^\bullet_{\mathbb{A}} \xrightarrow{\alpha} (K^\bullet_{\mathbb{A}\otimes\mathbb{Q}}, W) \xrightarrow{\beta} (K^\bullet_{\mathbb{C}}, W, F)$$

and the absolute Hodge cohomology is given by

$$R\Gamma_{\mathcal{H}}(K^\bullet) := \mathrm{Cone}^\bullet\big(K^\bullet_{\mathbb{A}} \oplus \widehat{W}_0 K^\bullet_{\mathbb{A}\otimes\mathbb{Q}} \oplus (\widehat{W}_0 \cap F^0)K^\bullet_{\mathbb{C}} \to {}'K^\bullet_{\mathbb{A}\otimes\mathbb{Q}} \oplus \widehat{W}_0\, {}'K^\bullet_{\mathbb{C}}\big)[-1]$$

where $\widehat{W}$ is the Deligne décalage filtration. According to Deligne [HodgeIII], there is a mixed Hodge complex associated to a smooth complex algebraic variety (to smooth simplicial varieties or posiblly singular complex algebraic varieties). An important result by Deligne [HodgeIII, 8.1.9], is that the cohomology of a mixed Hodge complex defines a mixed Hodge structure. In [Blo86b], Bloch constructed for X smooth, a cycle-class morphism

$$\mathrm{cl}_{q,n}\colon \mathrm{CH}^q(X,n) \to H^{2q-n}_{\mathcal{D}}(X,\mathbb{Z}(q))$$

where $H^{2q-n}_{\mathcal{D}}(X, \mathbb{Z}(q))$ is the Deligne-Beilinson cohomology. A. Goncharov [Gon95] and [Gon02] suggests that the regulator morphism should be induced by an explicitly defined morphism between these complexes. The **KLM-formula** is a morphism of complexes inducing the Bloch-Beilinson regulator morphism with rational coefficients. Kerr, Lewis and Müller-Stach [KLM06] gave such a morphism, using a cubical variant of Bloch's higher Chow groups to compute motivic cohomology, and a 3-term-complex to compute Deligne-Beilinson cohomology:

Theorem 0.0.2 [KLM06, 5.5] *There is a morphism of complexes*

$$\mathrm{reg}_X \colon \mathcal{Z}^p_{\mathbb{R}}(X, \bullet) \to C^{2p+\bullet}_{\mathcal{D}}(X, \mathbb{Q}(p)),$$

where the inclusion $\mathcal{Z}^p_{\mathbb{R}}(X, \bullet) \hookrightarrow \mathcal{Z}^p(X, \bullet)$ *is a quasi-isomorphism.*

The regulator is given in terms of integration of currents and describes the Bloch's cycle-morphism, further generalizing the Griffiths's Abel-Jacobi morphism.

0.5 Hodge realizations. On the other hand, if $\mathbf{MHC}^{\mathrm{p}}$ is the category of mixed Hodge complexes with polarizable cohomology [Bei86], there is a *realization functor*:

$$\mathrm{DM}_{\mathrm{gm}}(k) \to \mathbf{D}^b(\mathbf{MHS}^{\mathrm{p}})$$

from the category of geometric motives. This functor induces cycle class morphism. In [KL07], the authors generalize the Bloch's cycle class morphism to smooth quasi-projective varieties. In this generalization, Deligne cohomology is replaced by the absolute Hodge cohomology $H^*_{\mathcal{H}}$ which includes weights, and the regulator is the absolute Hodge cycle-class morphism

$$\mathrm{cl}_{\mathcal{H}} \colon \mathrm{CH}^p(X, n) \to H^{2p-n}_{\mathcal{H}}(X, \mathbb{Q}(p)).$$

Moreover, for a complete normal crossing divisor $Y \subset X$, they define an Abel-Jacobi morphism

$$H^{2p-n}_{\mathcal{M}}(Y, \mathbb{Q}(p)) \to H^{2p-n}_{\mathcal{H}}(Y, \mathbb{Q}(p))$$

where the motivic cohomology of Y is computed by semi-simplicial hyperresolutions in the spirit of Guillen-Navarro-Pascual-Puerta [GNPP88]. This suggested that the KLM-formula can be extended to smooth simplicial varieties, o more generally, to singular varieties, using semi-simplicial hyperresolutions of varieties over the complex numbers.

Motivic cohomology and regulators for singular varieties

In the case of singular varieties, the higher Chow groups fail as motivic cohomology, as these form a motivic Borel-Moore homology theory (as in algebraic topology). In this case, pull-backs behave better for motivic cohomology than for higher Chow groups. However, much of the construction used in the case of normal crossing divisors extends to singular varieties, such as motivic cohomology, absolute Hodge cohomology, regulators and Abel-Jacobi morphisms extend directly using simplicial hypercovers.

0.6 Hanamura's motivic cohomology. Using the extension criterion of Guillén-Navarro [GNA02], Hanamura [Han00] extends the definition of motivic cohomology to the case of singular varieties, and other theories starting from higher Chow groups. This is a contravariant functor from quasi-projective varieties to abelian groups

$$X \ \mapsto \ \mathrm{CHC}^r(X,m);$$

which coincides with $\mathrm{CH}^r(X,m)$ for X smooth. The procedure is via cubical hyperresolutions (analogous to Deligne hypercoverings) [GNPP88]. For a singular quasi-projective variety X, consider its cubical hyperresolution $a\colon X_\bullet \to X$. This is a semi-simplicial scheme of the form

$$\cdots \begin{smallmatrix}\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow\end{smallmatrix} X_2 \ \begin{smallmatrix}\xrightarrow{d_2}\\ \xrightarrow{d_1}\\ \xrightarrow[d_0]{}\end{smallmatrix} \ X_1 \ \begin{smallmatrix}\xrightarrow{d_1}\\ \xrightarrow[d_0]{}\end{smallmatrix} \ X_0 \xrightarrow{a} X$$

consisting of smooth quasi-projective varieties X_p, with an augmentation a to X, satisfying certain conditions with the face morphisms. The cubical hyperresolution has the property that the cohomology of X can be computed by the cohomology of $X_\bullet$, the cohomological descent property. Applying the technique of Bloch's cycle complex to hyperresolutions, we obtain a double complex

$$0 \longrightarrow \mathcal{Z}^r(X_0,\bullet) \xrightarrow{d} \mathcal{Z}^r(X_1,\bullet) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{Z}^r(X_p,\bullet) \longrightarrow \cdots$$

where $\mathcal{Z}^r(X_p,\bullet)$ is the Bloch's cycle complex of X_p, and the (horizontal) differentials are given by $d = \sum(-1)^i d_i^*$. Let $\mathcal{Z}^r(X_\bullet)^*$ be its total complex. **Hanamura's motivic cohomology** is defined as:

$$\mathrm{CHC}^r(X,m) = H_m\big(\mathcal{Z}^r(X_\bullet)^*\big).$$

The definition is independent, up to isomorphism, of the choice of hyperresolution. Naturally there is a spectral sequence associated with this double complex, the Hanamura spectral sequence

$$E_1^{p,q}(r) := \mathrm{CH}^r(X_p, -q) \Rightarrow \mathrm{CHC}^r(X, p-q).$$

Many of the properties for smooth varieties can be inherited to singular varieties, such as homotopy invariance, Mayer-Vietoris, etc. The **motivic cohomology of Friedlander-Voevodsky** of a singular variety is defined in terms of Suslin-Friedlander motivic complex, using the cdh-hypercohomology [FV00]. An important result is that Hanamura's motivic cohomology agrees with the motivic cohomology of Friedlander-Voevodsky, as suggested by Hanamura in [Han14]:

Theorem 0.0.3 *Let X be a quasi-projective variety over a field k that admits resolution of singularities. Then, there exists an isomorphism*

$$\mathrm{CHC}^r(X, m) \cong H_{\mathcal{M}}^{2r-m}(X, \mathbb{Q}(r)),$$

where $H_{\mathcal{M}}^{2r-m}(X, \mathbb{Q}(r))$ is Friedlander-Voevodsky motivic cohomology.

The reason for this isomorphism is that for smooth varieties, there is a quasi-isomorphism $\mathbb{Z}^{SF}(r)[2r] \to \mathcal{Z}^r(- \times \mathbb{A}^r, \bullet)$ of complexes of Zariski sheaves between Suslin-Friedlander complex and Bloch's cycle complex, see [MVW06, Theo. 19.8], and using spectral sequences and cdh-descent for motivic cohomology.

0.7 A regulator morphism for Hanamura's motivic cohomology. As noted by Kerr and Lewis, the regulator morphism admits an extension to Hanamura groups for an arbitrary variety. The model given by the cycle class morphism for a normal crossing divisor, allows to extend the general construction of the regulator morphism from Hanamura's cohomological Chow groups (motivic cohomology) to absolute Hodge cohomology:

Theorem 0.0.4 *There is a morphism of (double) complexes in the derived category $\mathcal{Z}^{p,q}(r) \to \mathcal{H}^{p,q}(r)$ given by the KLM-formula, with a morphism of total complexes $\mathcal{Z}^r(X_\bullet)^* \to \mathcal{H}_{X_\bullet}(r)$. This regulator morphism induces a cycle-class morphism on the total cohomologies*

$$H_{\mathcal{M}}^{2r-*}(X, \mathbb{Q}(r)) \cong \mathrm{CHC}^r(X, *) \to H_{\mathcal{H}}^{2r-*}(X, \mathbb{Q}(r)).$$

Such a morphism coincides, when X is smooth, with the KLM-regulator.

The main idea is to use a cubical hyperresolution $X_\bullet \to X$ to define the complexes that give us both cohomologies, and then use the KLM formula in the smooth case. This morphism generalizes the Bloch cycle-class morphism.

Outline of the work

In the first chapter, we give an overview of the cohomological machinery, necessary for the following chapters. We recall the notion of spectral sequences, important to define a mixed Hodge structure on a complex algebraic variety. We also review the definitions of Deligne cohomology and homology, important for defining the regulator morphism and the Abel-Jacobi morphism.

In Chapter 2, we recall the construction of the Abel-Jacobi morphism (KLM-formula). This is a morphism form higher Chow groups (motivic cohomology) to Deligne cohomology, in the smooth and projective case. This is a morphism from higher Chow groups to Deligne cohomology, given in terms of an explicit morphism between complexes that define such cohomologies. In the smooth and quasi-projective case, we substitute the Deligne cohomology with the absolute Hodge cohomology.

In the chapter 3, we review two definitions of motivic cohomology for singular varieties. The first, Hanamura's motivic cohomology, given in terms of cubical hyperresolutions. The other, Friedlander-Voevodsky motivic cohomology, given as a cdh-cohomology. Using the descent criterion for the cdh-topology and induction in the length of the cubical hyperresolution, we establish an isomorphism of comparison between such cohomologies.

In the chapter 4, we define the mixed Hodge complex associate to a singular quasi-projective variety, and then we consider its associated absolute Hodge cohomology. In this case, the regulator morphism is given in terms of the complexes that define the cohomology groups. The theorem (4.3.1) gives a morphism between spectral sequences, this is the regulator morphism in the singular case.

In the last chapter, we give some calculations of motivic cohomology using Hanamura's technique, for some special cases.

Chapter 1

Cohomological preliminaries

Let $\mathcal{A}$ be an abelian category. A (cohomological) **complex** is a sequence of objects $A^i \in \mathcal{A}$ together with differentials $d^i\colon A^i \to A^{i+1}$ such that the composition $d^{i+1} \circ d^i = 0$ for all $i \in \mathbb{Z}$. The **category of complexes** is denoted by $\mathbf{C}(\mathcal{A})$. We write $\mathbf{C}^+(\mathcal{A})$, $\mathbf{C}^-(\mathcal{A})$ and $\mathbf{C}^b(\mathcal{A})$ for the full subcategories of complexes bounded below, bounded above and bounded respectively. If $A^\bullet \in \mathbf{C}(\mathcal{A})$ is a complex, we define the **shifted complex** $A^\bullet[q]$ with

$$A^\bullet[q]^i = A^{i+q} \qquad \text{and} \qquad d^i_{A^\bullet[q]} = (-1)^q d^{i+q}_{A^\bullet}.$$

If $f\colon (A^\bullet, d) \to (B^\bullet, d)$ is a morphism of complexes, the **cone complex** is defined by the formula

$$\mathrm{Cone}(A^\bullet \xrightarrow{f} B^\bullet) := A^\bullet[1] \oplus B^\bullet,$$

together with the differentials

$$\begin{aligned} d^i_{\mathrm{Cone}(f)}\colon A^{i+1} \oplus B^i &\to A^{i+2} \oplus B^{i+1} \\ (a,b) &\mapsto \big(-d_A^{i+1}(a), f^{i+1}(a) + d^i_B(b)\big). \end{aligned}$$

This gives a short exact sequence

$$0 \to B^\bullet \to \mathrm{Cone}^\bullet(f) \to A^\bullet[1] \to 0.$$

Denote by $\mathbf{K}(\mathcal{A})$ the corresponding **homotopy category** where the objects are the same and the morphisms are homotopy classes of morphisms of complexes. This category is always triangulated with translation functor $[1]\colon \mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{A})$, and the class of distinguished triangles are those homotopy equivalent to the **standard distinguished triangle**

$$A^\bullet \xrightarrow{f} B^\bullet \to \mathrm{Cone}^\bullet(f) \to A^\bullet[1]$$

for some morphism of complexes f. Let $\mathbf{D}(\mathcal{A})$ be the induced **derived category**, where the objects are the same as in $\mathbf{K}(\mathcal{A})$ and morphisms are obtained by localising $\mathbf{K}(\mathcal{A})$ with respect to the class of quasi-isomorphisms[1]. The derived category is also triangulated where the class of distinguished triangles are those isomorphic in $\mathbf{D}(\mathcal{A})$ to a distinguished triangle in $\mathbf{K}(\mathcal{A})$.

1.1. The total complex and filtrations

Let $\mathcal{A}$ be an additive category. A double complex $(A^{\bullet,\bullet}, d_1^{\bullet,\bullet}, d_2^{\bullet,\bullet}) \in \mathbf{C}(\mathcal{A})$ is a diagram of objects $A^{p,q} \in \mathcal{A}$ for $p, q \in \mathbb{Z}$ with two differentials

$$d_1^{p,q} \colon A^{p,q} \to A^{p+1,q} \qquad \text{and} \qquad d_2^{p,q} \colon A^{p,q} \to A^{p,q+1}$$

where $(A^{\bullet,q}, d_1^{\bullet,q})$ and $(A^{p,\bullet}, d_2^{p,\bullet})$ are complexes, and the squares

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
\cdots \arrow[r] & A^{p-1,q+1} \arrow[r] \arrow[u] & A^{p,q+1} \arrow[r, "d_1^{p,q+1}"] \arrow[u] & A^{p+1,q+1} \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & A^{p-1,q} \arrow[r] \arrow[u] & A^{p,q} \arrow[r, "d_1^{p,q}"'] \arrow[u, "d_2^{p,q}"] & A^{p+1,q} \arrow[r] \arrow[u, "d_2^{p+1,q}"'] & \cdots \\
\cdots \arrow[r] & A^{p-1,q-1} \arrow[r] \arrow[u] & A^{p,q-1} \arrow[r] \arrow[u] & A^{p+1,q-1} \arrow[r] \arrow[u] & \cdots \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

commute for all $p, q \in \mathbb{Z}$, i.e. $d_2^{p+1,q} \circ d_1^{p,q} = d_1^{p,q+1} \circ d_2^{p,q}$ (there is another version of the double complex with the property that squares are anti-commutative, i.e. $d_2^{p+1,q} \circ d_1^{p,q} + d_1^{p,q+1} \circ d_2^{p,q} = 0$, in some contexts this convention is better, as we will see below). The associated **total complex** (or **simple complex**) $\mathrm{Tot}(A^{\bullet,\bullet})$ is defined as

$$\mathrm{Tot}(A^{\bullet,\bullet})^n = \bigoplus_{p+q=n} A^{p,q} \qquad \text{and} \qquad d^n_{\mathrm{Tot}(A^{\bullet,\bullet})} = \sum_{p+q=n} \big(d_2^{p,q} + (-1)^q d_1^{p,q}\big).$$

[1] In an abelian category $\mathcal{A}$, a morphism of complexes $f^\bullet \colon A^\bullet \to B^\bullet$ is called **quasi-isomorphism** if $H^q(f) \colon H^q(A^\bullet) \to H^q(B^\bullet)$ is an isomorphism for all $q \in \mathbb{Z}$.

In this case, we have the identity $d_{\mathrm{Tot}}^{n+1} \circ d_{\mathrm{Tot}}^{n} = 0$. For a triple complex $A^{\bullet,\bullet,\bullet}$, the associated total complex is the complex with terms

$$\mathrm{Tot}(A^{\bullet,\bullet,\bullet})^n = \bigoplus_{p+q+r=n} A^{p,q,r}$$

with differential

$$d^n_{\mathrm{Tot}(A^{\bullet,\bullet,\bullet})} = \sum_{p+q+r=n} \big(d_1^{p,q,r} + (-1)^p d_2^{p,q,r} + (-1)^{p+q} d_3^{p,q,r}\big).$$

With this definition, it is easy to see that:

$$\mathrm{Tot}(A^{\bullet,\bullet,\bullet}) = \mathrm{Tot}(\mathrm{Tot}_{1,2}(A^{\bullet,\bullet,\bullet})) = \mathrm{Tot}(\mathrm{Tot}_{2,3}(A^{\bullet,\bullet,\bullet})).$$

This means that we can either first combine the first two variables and then combine the sum of those with the last, or we can first combine the last two variables and then combine the first with the sum of the last two.

Example 1.1.1 The construction of the cone complex is a special case of the total complex of a double complex given by a morphism of complexes. Let $h\colon A^\bullet \to B^\bullet$ be a morphism, we define the double complex $C^{\bullet,\bullet}$ by

$$\begin{array}{ccccccccc}
B^\bullet & = C^{\bullet,(0)}: & B^{0,(0)} & \longrightarrow & B^{1,(0)} & \longrightarrow & B^{2,(0)} & \longrightarrow & \cdots \\
{\scriptstyle h}\uparrow & \quad {\scriptstyle h^{\bullet,-1}}\uparrow & {\scriptstyle h^{0,-1}}\uparrow & & {\scriptstyle h^{1,-1}}\uparrow & & {\scriptstyle h^{2,-1}}\uparrow & & \\
A^\bullet & = C^{\bullet,(-1)}: & A^{0,(-1)} & \longrightarrow & A^{1,(-1)} & \longrightarrow & A^{2,(-1)} & \longrightarrow & \cdots
\end{array}$$

Then, the cone complex of h is $\mathrm{Cone}^\bullet(h) = \mathrm{Tot}(C^{\bullet,\bullet})$.

Notation. For simplicity, we will use $\mathbf{s}(A^{\bullet,\bullet})$ to denote the total complex (or simple complex) of $A^{\bullet,\bullet}$, instead of $\mathrm{Tot}(A^{\bullet,\bullet})$.

1.1.2 Let $A^\bullet$ be a complex in an additive category $\mathcal{A}$:

1. The **trivial filtration** $\sigma^{\geq p} A^\bullet$ is a decreasing filtration given by

$$\sigma^{\geq p} A^\bullet := A^{\bullet \geq p} := \{0 \to 0 \to \cdots \to 0 \to A^p \to A^{p+1} \to \cdots\}.$$

The quotient $A^\bullet / \sigma^{\geq p} A^\bullet$ is given by

$$\sigma^{<p} A^\bullet := A^{\bullet < p} := \{\cdots \to A^{p-2} \to A^{p-1} \to 0 \to 0 \to \cdots\}.$$

The p-graded piece is $\mathrm{Gr}^p_\sigma A^\bullet = A^p[-p]$ in degree p and zero elsewhere.

2. The **canonical filtration** $\tau_{\leq p}A^\bullet$ is an increasing filtration

$$\tau_{\leq p}A^\bullet := \{\cdots \to A^{p-2} \to A^{p-1} \to \ker(d^p) \to 0 \to \cdots\}$$

such that the quotient $A^\bullet/\tau_{\leq p}A^\bullet$ is given by

$$\tau_{>p}A^\bullet := \{\cdots \to 0 \to A^p/\ker(d^p) \to A^{p+1} \to \cdots\}.$$

The p-graded piece of the filtration is the complex

$$\tau_{\leq p}A^\bullet/\tau_{\leq p-1}A^\bullet := \{0 \to A^{p-1}/\ker(d^{p-1}) \to \ker(d^p) \to 0\}$$

quasi-isomorphic to the complex $H^p(A^\bullet)$ in degree p, i.e. $\mathrm{Gr}^\tau_p A^\bullet \cong H^p(A^\bullet)[-p]$.

1.2. Spectral sequences of filtered complexes

In order to construct spectral sequences, filtrations on complexes are important. Spectral sequences are a powerful tool for working with cohomology. This technique permits to endow with a mixed Hodge structure the cohomology groups, such that each page carries a natural pure Hodge structure of certain weight and all diferentials between pages are morphisms of Hodge structures [HodgeII], [GH78]. Let $\mathcal{A}$ be an abelian category.

Definition 1.2.1 Let $(A^\bullet, d) = \{A^0 \xrightarrow{d} A^1 \xrightarrow{d} A^2 \xrightarrow{d} \cdots\}$ be a bounded complex in $\mathcal{A}$. A **decreasing filtered complex**[2], is a family of subcomplexes of $(A^\bullet, d)$ denoted by:

$$A^\bullet = F^0A^\bullet \supset F^1A^\bullet \supset \cdots \supset F^NA^\bullet \supset F^{N+1}A^\bullet = \{0\}.$$

The **associated graded complex** to a filtered complex $(F^\nu A^\bullet, d)$ is:

$$\mathrm{Gr}_F A^\bullet = \bigoplus_{\nu\geq 0} \mathrm{Gr}^\nu_F A^\bullet = \bigoplus_{\nu\geq 0} \frac{F^\nu A^\bullet}{F^{\nu+1}A^\bullet}$$

with the induced differentials. The filtration F^ν also induces a filtration $F^\nu H^*(A^\bullet)$ on the cohomology via:

$$F^\nu H^p(A^\bullet) := F^\nu A^p_{d-\text{closed}} \;/\; F^\nu \cap dA^{p-1}.$$

[2] In the same way, a **increasing filtration** $W_\bullet$ on $A^\bullet$ is a family of subcomplexes of $A^\bullet$ with $\cdots \subset W_{p-1}A^\bullet \subset W_pA^\bullet \subset W_{p+1}A^\bullet \subset \cdots$. We use the notation $F^\bullet$ for decreasing filtrations, and $W_\bullet$ for increasing filtrations.

This gives

$$H^p(A^\bullet) = F^0H^p(A^\bullet) \supset F^1H^p(A^\bullet) \supset \cdots \supset F^\nu H^p(A^\bullet) \supset F^{\nu+1}H^p(A^\bullet) \supset \cdots$$

The **associated graded cohomology** is

$$\mathrm{Gr}_F H^*(A^\bullet) = \bigoplus_{\nu,p\geq 0} \mathrm{Gr}_F^\nu H^p(A^\bullet) = \bigoplus_{\nu,p\geq 0} \frac{F^\nu H^p(A^\bullet)}{F^{\nu+1}H^p(A^\bullet)}.$$

Denote by $\mathbf{C}^+(\mathbf{F}\mathcal{A})$ the category of bounded below filtered complexes on $\mathcal{A}$. An object in this category is of the form $(A^\bullet, F)$ with F a filtration on $A^\bullet$, and a **morphism of filtered complexes** $f\colon (A^\bullet, F) \to (B^\bullet, F)$ is a morphism of complexes $f\colon A^\bullet \to B^\bullet$ such that $f(F^pA^\bullet) \subset F^pB^\bullet$. If F is a decreasing filtration (resp. W an increasing filtration), a **shifted filtration** $F[n]$ by an integer n is defined as

$$(F[n])^pA = F^{p+n}A, \qquad (W[n])_pA = W_{p-n}A.$$

To define spectral sequences we consider decreasing filtrations; statements for increasing filtrations are deduced by the change of indices $W_\bullet := F^{-\bullet}$. We say that a filtration is **finite** if there exist integers $m > n$ with $F^m = 0$ and $F^n = A^\bullet$. The filtration $F^\bullet$ is **biregular** if it is a finite filtration on each component of $A^\bullet$. The filtrations that we will consider will always be biregular. Given a filtered complex, the filtrations give us a way to approximate its cohomology. The principal aim of spectral sequences is to compute $\mathrm{Gr}_F^p H^*(A^\bullet)$.

Definition 1.2.2 A **spectral sequence** is a sequence $\{E_r, d_r\}_{r\geq 0}$ of bi-graded groups

$$E_r = \bigoplus_{p,q\geq 0} E_r^{p,q}$$

together with differentials

$$d_r\colon E_r^{p,q} \to E_r^{p+r,q-r+1}, \qquad d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0$$

such that $H^*(E_r) = E_{r+1}$.

Proposition 1.2.3 ([GH78]) *Let $(A^\bullet, d, F^\bullet)$ be a filtered complex. Then there exists a spectral sequence $\{E_r\}$ with*

$$\begin{aligned} E_0^{p,q} &= \frac{F^pA^{p+q}}{F^{p+1}A^{p+q}} =: \mathrm{Gr}_F^p A^{p+q}, \\ E_1^{p,q} &= H^{p+q}(\mathrm{Gr}_F^p A^\bullet), \\ E_\infty^{p,q} &= \mathrm{Gr}_F^p\big(H^{p+q}(A^\bullet)\big). \end{aligned}$$

The last statement is usually written: $E_r^{p,q} \Rightarrow H^{p+q}(A^\bullet)$*, and we say that the spectral sequence converges to* $H^*(A^\bullet)$*.*

Proof. The $E_0^{p,q}$ term has been defined:

$$\begin{array}{ccccc} E_0^{p,q-1} & \xrightarrow{d_0^{p,q-1}} & E_0^{p,q} & \xrightarrow{d_0^{p,q}} & E_0^{p,q+1} \\ \| & & \| & & \| \\ \frac{F^pA^{p+q-1}}{F^{p+1}A^{p+q-1}} & \xrightarrow{d} & \frac{F^pA^{p+q}}{F^{p+1}A^{p+q}} & \xrightarrow{d} & \frac{F^pA^{p+q+1}}{F^{p+1}A^{p+q+1}} \end{array}$$

where d_0 is induced by d. By definition, $E_1^{p,q}$ is the cohomology of $E_0^{p,q}$:

$$\begin{aligned} E_1^{p,q} = \frac{\operatorname{Ker}(d_0^{p,q})}{\operatorname{Im}(d_0^{p,q-1})} &= \frac{\{a \in F^pA^{p+q} \mid d(a) \in F^{p+1}A^{p+q+1}\}}{d(F^pA^{p+q-1}) + F^{p+1}A^{p+q}} \\ &= H^{p+q}\left(\frac{F^pA^\bullet}{F^{p+1}A^\bullet}\right) \\ &= H^{p+q}(\mathrm{Gr}_F^p A^\bullet). \end{aligned}$$

The differentials in the $E_1^{p,q}$ term are given by

$$\begin{array}{ccc} E_1^{p-1,q} & = & \frac{\{a\in F^{p-1}A^{p+q-1} \mid d(a)\in F^pA^{p+q}\}}{\{d(F^{p-1}A^{p+q-2})+F^pA^{p+q-1}\}} \\ \downarrow{\scriptstyle d_1^{p-1,q}} & & \downarrow{\scriptstyle d} \\ E_1^{p,q} & = & \frac{\{a\in F^pA^{p+q} \mid d(a)\in F^{p+1}A^{p+q+1}\}}{\{d(F^pA^{p+q-1})+F^{p+1}A^{p+q}\}} \\ \downarrow{\scriptstyle d_1^{p,q}} & & \downarrow{\scriptstyle d} \\ E_1^{p+1,q} & = & \frac{\{a\in F^{p+1}A^{p+q+1} \mid d(a)\in F^{p+2}A^{p+q+2}\}}{\{d(F^{p+1}A^{p+q})+F^{p+2}A^{p+q+1}\}} \end{array}$$

Then, the term $E_2^{p,q}$ is:

$$E_2^{p,q} := \frac{\operatorname{Ker}(d_1^{p,q})}{\operatorname{Im}(d_1^{p-1,q})} = \frac{\{a \in F^pA^{p+q} \mid d(a) \in F^{p+2}A^{p+q+1}\}}{d(F^{p-1}A^{p+q-1}) + F^{p+1}A^{p+q}}.$$

In general, we define the term $E_r^{p,q}$ with its differential:

$$\begin{array}{rcl} E_r^{p,q} & := & \frac{\{a\in F^pA^{p+q} \mid d(a)\in F^{p+r}A^{p+q+1}\}}{d(F^{p-r+1}A^{p+q-1})+F^{p+1}A^{p+q}} \\ & & \downarrow{\scriptstyle d_r^{p,q}} \\ E_r^{p+r,q-r+1} & := & \frac{\{a\in F^{p+r}A^{p+q+1} \mid d(a)\in F^{p+2r+1}A^{p+q+2}\}}{d(F^pA^{p+q})+F^{p+r+1}A^{p+q+1}}. \end{array}$$

For r sufficiently large

$$\begin{aligned} E_r^{p,q} = E_\infty^{p,q} &= \frac{\{a \in F^p A^{p+q} \mid d(a) = 0\}}{d(A^{p+q-1}) + F^{p+1} A^{p+q}} \\ &= \frac{F^p H^{p+q}(A^\bullet)}{F^{p+1} H^{p+q}(A^\bullet)} \\ &=: \operatorname{Gr}_F^p H^{p+q}(A^\bullet). \end{aligned}$$

A similar computation gives us that $H^*(E_r^{p,q}) \cong E_{r+1}^{p,q}$. $\square$

Remark 1.2.4 The sum of E_r's is commonly called the r-th page. Then, each d_r is a morphism on the r-th page of the spectral sequence. Although in this work we always consider spectral sequences of (filtered) complexes, all the definitions make sense in any abelian category.

Example 1.2.5 One of the principal examples is the spectral sequence associated to a (bounded) double complex

$$A^{\bullet,\bullet} = \bigoplus_{p,q \geq 0} A^{p,q}, \quad d\colon A^{p,q} \to A^{p+1,q}, \quad \delta\colon A^{p,q} \to A^{p,q+1},$$

with $d^2 = \delta^2 = 0$ and $d\delta + \delta d = 0$ (a double complex with anticommutative squares). For simplicity suppose that $A^{\bullet,\bullet}$ is first quadrant, i.e. $A^{p,q} = 0$ for $p < 0$ or $q < 0$. The associated total complex is

$$\mathbf{s}(A^{\bullet,\bullet})^n = \bigoplus_{p+q=n} A^{p,q}, \quad D = d + \delta.$$

There are two descending filtrations on $(\mathbf{s}A^{\bullet,\bullet}, D)$ given by

$$'F^\nu \mathbf{s}A^n := \bigoplus_{p+q=n, p \geq \nu} A^{p,q}$$

$$''F^\nu \mathbf{s}A^n := \bigoplus_{p+q=n, q \geq \nu} A^{p,q}.$$

These filtrations define two spectral sequences:

$$'E_r^{p,q} \Rightarrow H_D^{p+q}(\mathbf{s}A^{\bullet,\bullet})$$

$$''E_r \Rightarrow H_D^{p+q}(\mathbf{s}A^{\bullet,\bullet}).$$

For the first one

$$'E_0^{p,q} = \frac{A^{p,q} \oplus A^{p+1,q-1} \oplus \cdots}{A^{p+1,q-1} \oplus A^{p+2,q-2} \oplus \cdots} \cong A^{p,q}$$

with differential d_0 induced by $D = d + \delta$. In this case, $d_0 = \partial$ and

$$'E_1^{p,q} = H_D^{p+q}(\mathrm{Gr}^p_{'F}\mathbf{s}A^{\bullet,\bullet}) = H^q_\partial(A^{p,\bullet}),$$

the differential d_1 comes from $D = d+\delta$ on $'E_1$. In this case, $d = d_1$ beacuse $\partial = 0$ on $'E_1$ and we have:

$$'E_2^{p,q} = H^*('E_1^{p,q}, d_1) \cong H^p_d\big(H^q_\partial(A^{\bullet,\bullet})\big).$$

In the same way, we have that $''E_2^{p,q} \cong H^p_\partial\big(H^q_d(A^{\bullet,\bullet})\big)$. In summary, there are two spectral sequences associated to a double complex which both converge to the cohomology of the total complex.

1.2.6 The décalage filtration. Let $f\colon (A^\bullet, F) \to (B^\bullet, F')$ be a morphism of filtered complexes. It induces a morphism of complexes between associated grades $\mathrm{Gr}(f)\colon \mathrm{Gr}_F A^\bullet \to \mathrm{Gr}_{F'} B^\bullet$, and induces a morphism between associated spectral sequences

$$f_r\colon E_r(F) \to E_r(F').$$

We say that f is a **filtered quasi-isomorphism** if f_r are isomorphisms for $r \geq 1$. In order to endow the cohomology of a mixed Hodge complex with a mixed Hodge structure, Deligne introduced the *décalage functor* in [HodgeII]:

Definition 1.2.7 Let $(A^\bullet, W_\bullet)$ be a filtered complex. The **décalage filtration** is a functor $\mathrm{Dec}\colon \mathbf{C}^+(\mathbf{F}\mathcal{A}) \to \mathbf{C}^+(\mathbf{F}\mathcal{A})$ defined by

$$(\mathrm{Dec}\ W)_\ell A^m := \ker\Big\{ W_{\ell-m}A^m \xrightarrow{d_A} \frac{A^{m+1}}{W_{\ell-m-1}A^{m+1}} \Big\}.$$

Theorem 1.2.8 ([HodgeII, Prop. 1.3.4]) *The canonical morphisms*

$$E_0^{p,q}(\mathrm{Dec}\ W) \to E_1^{2p+q,-p}(W)$$

are quasi-isomorphisms of bigraded complexes. The induced morphisms

$$E_r^{p,q}(\mathrm{Dec}\ W) \to E_{r+1}^{2p+q,-p}(W)$$

are isomorphisms for $r \geq 1$.

1.3. Formal aspects of Mixed Hodge Structures

We give a brief overview of Hodge structures. We start with a quick review of the notions of pure Hodge structures and mixed Hodge structures. The Deligne's theorem asserts the existence of a functorial mixed Hodge structure on the cohomology of an arbitrary complex algebraic variety [HodgeI], [HodgeII] and [HodgeIII]. Also we review some results on extensions of mixed Hodge structures [PS08].

Definition 1.3.1 Let $\mathbb{A} \subset \mathbb{R}$ be a subring. A **pure $\mathbb{A}$-Hodge structure** of weight ℓ consists of a finitely generated $\mathbb{A}$-module H with a direct sum decomposition, called the **Hodge decomposition**

$$H_{\mathbb{C}} := H \otimes_{\mathbb{A}} \mathbb{C} = \bigoplus_{p+q=\ell} H^{p,q}, \quad \text{with} \quad \overline{H^{p,q}} = H^{q,p}.$$

Equivalently, $H_{\mathbb{C}}$ admits a decreasing filtration $F^{\bullet}$, the **Hodge filtration** such that

$$H_{\mathbb{C}} = F^r H_{\mathbb{C}} \bigoplus \overline{F^{\ell-r+1} H_{\mathbb{C}}}.$$

The relation is given by

$$F^r H_{\mathbb{C}} = \bigoplus_{p+q=\ell,\ p\geq r} H^{p,q}, \quad \text{and} \quad H^{p,q} = F^p H_{\mathbb{C}} \bigcap \overline{F^q H_{\mathbb{C}}}.$$

Example 1.3.2 Let X be a smooth, projective variety over $\mathbb{C}$. The prototype of a pure Hodge structure of weight ℓ, is the ℓ-th cohomology group $H^{\ell}(X, \mathbb{Z})$. In this case, we have the Hodge decomposition theorem:

$$H^{\ell}_{\mathrm{DR}}(X, \mathbb{C}) = \bigoplus_{p+q=\ell} H^{p,q}(X).$$

This construction gives a functor:

$$H^* \colon \mathbf{SmProj}(\mathbb{C}) \to \mathbf{HS}$$

where $\mathbf{SmProj}(\mathbb{C})$ is the category of smooth, projective varieties over $\mathbb{C}$.

Example 1.3.3 The **Hodge structure of Tate type** $\mathbb{Q}(r) := (2\pi i)^r \mathbb{Q}$ is a $\mathbb{Q}$-Hodge structure of weight $-2r$ and of pure Hodge type $(-r, -r)$, the decomposition is given by $\mathbb{C} = \mathbb{C}^{-r,-r}$. The **Lefschetz structure** is $\mathbb{Q}(-1)$.

Example 1.3.4 Let X be a smooth, projective variety over $\mathbb{C}$. The r-th **Tate twist** of $H^\ell(X,\mathbb{Q})$ is defined as

$$H^\ell(X,\mathbb{Q}(r)) := H^\ell(X,\mathbb{Q}) \otimes \mathbb{Q}(r),$$

this is a $\mathbb{Q}$-Hodge structure of weight $\ell - 2r$.

A **graded pure Hodge structure** is a finite direct sum of pure Hodge structures, possibly of different weights. A **morphism of Hodge structures** $f\colon V \to W$ is a $\mathbb{Q}$-linear map which induces $f_{\mathbb{C}}\colon V_{\mathbb{C}} \to W_{\mathbb{C}}$ having the property that it preserves type, i.e. $f_{\mathbb{C}}(V^{p,q}) \subset W^{p,q}$ for all p and q. The category of pure Hodge structures admits direct sums, tensor products, Hom's and duals. The category of Hodge structures is an abelian category which we denote by **HS** [PS08]. In general, the category **HS** is not semi-simple; this is remedied if we restrict ourselves to Hodge structures that admit a *polarization*.

Definition 1.3.5 Let H be a Hodge structure of weight ℓ. A **polarization** of H is a nonsingular, bilinear form $S\colon H_{\mathbb{C}} \otimes H_{\mathbb{C}} \to \mathbb{C}$ wich is defined over $\mathbb{Q}$, such that:

i) $S(x,y) = (-1)^\ell S(y,x)$.

ii) $S(H^{p,q}, H^{r,s}) = 0$ unless $p = s, q = r$.

iii) $i^{p-q} S(x,\overline{y})$ is a hermitian positive-definite bilinear form on $H^{p,q}$.

A Hodge structure that admits a polarization is said to be **polarizable**. The cohomology groups of non-singular and projective complex varieties are endowed with polarizable Hodge structures. The category of polarizable Hodge structures is semi-simple [PS08, Cor. 2.12].

To extend these ideas to quasi-projective and singular varieties, we give the main definition of this section, that of a mixed Hodge structure.

Definition 1.3.6 An $\mathbb{A}$-**mixed Hodge structure** consists of the following:

– a finitely generated $\mathbb{A}$-module $H_{\mathbb{A}}$,

– a finite increasing filtration $W_\bullet$ on $H_{\mathbb{A}} \otimes \mathbb{Q} := H_{\mathbb{A}} \otimes_{\mathbb{Z}} \mathbb{Q}$, called the **weight filtration**,

– a finite descending filtration $F^\bullet$ on $H_{\mathbb{C}} := H_{\mathbb{A}} \otimes \mathbb{C}$, called the **Hodge filtration**, such that $\{F^r\}$ induces a (pure) HS of weight ℓ on $\mathrm{Gr}_\ell^W := W_\ell/W_{\ell-1}$, i.e. the graged piece $\mathrm{Gr}_\ell^W H_{\mathbb{A}} \otimes \mathbb{Q}$ with the filtration $F^r \mathrm{Gr}_\ell^{W\otimes\mathbb{C}} H_{\mathbb{C}}$ is in $\mathbf{HS}_{(\ell)}$.

A **morphism of mixed Hodge structures** is a morphism $f\colon H \to H'$ that tensored with $\mathbb{Q}$, is compatible with the weight filtration $W_\bullet$, and when is tensored with $\mathbb{C}$, is compatible with the Hodge filtration $F^\bullet$. We denote the category of mixed Hodge structures by **MHS**. The category of mixed Hodge structures has natural direct sums, tensor products, duals and internal Hom's. This makes the category **MHS** abelian and rigid [HodgeII, Theo. 2.3.5]. For any morphism $f\colon H \to H'$ of **MHS**, we have an induced morphism $\mathrm{Gr}_\ell^W(f)$ of pure Hodge structures of weight ℓ. The functor $\mathrm{Gr}_\ell^W : \mathbf{MHS} \to \mathbf{HS}_{(\ell)}$ is exact. We say that a mixed Hodge structure H is **polarizable** if each of the graded pieces $\mathrm{Gr}_\ell^W H$ is pure polarizable.

A fundamental result of Deligne [HodgeII, HodgeIII], shows that for any complex variety X, the cohomology $H^\ell(X,\mathbb{Z})$ carries a canonical and functorial mixed Hodge structure. This is the main theorem on the existence of MHS:

Theorem 1.3.7 [HodgeII, HodgeIII] *Let X be a complex algebraic variety and $Y \subset X$ a closed subvariety (possibly empty). Then $H^\ell(X,Y)$ has a mixed Hodge structure which is functorial in the sense that if $f\colon (X,Y) \to (X',Y')$ is a morphism of pairs, the induced morphism $f^*\colon H^\ell(X',Y') \to H^\ell(X,Y)$ is a morphism of mixed Hodge structures. Moreover, if X is smooth and projective, the corresponding mixed Hodge structure on $H^\ell(X)$ is the classical pure Hodge structure of weight ℓ.*

The **Hodge filtration** $F^\bullet$ on the cohomology $H^m(X,\mathbb{C})$ satifies

$$H^m(X,\mathbb{C}) = F^0 \supseteq \cdots \supseteq F^{m+1} = 0.$$

Moreover, for $m \geq d = \dim_{\mathbb{C}}(X)$ we also have that $F^{d+1} = 0$.

Weights on cohomology. According to Deligne's Theorem (1.3.7), for a quasi-projective X, there is a functorial increasing filtration $W_\bullet$ on the rational cohomology $H^m(X,\mathbb{Q})$, the **weight filtration**:

$$0 = W_{-1}H^m(X) \subset W_0 H^m(X) \subset \cdots \subset W_{2m}H^m(X) = H^m(X).$$

This filtration relates the cohomology of the variety with cohomologies of smooth, projective varieties via the quotients

$$\mathrm{Gr}_\ell^W H^m(X) := W_\ell H^m(X)/W_{\ell-1}H^m(X)$$

where for each ℓ, the graded piece has a pure Hodge structure of weight ℓ. This weight filtration has the following presentation [Dur83]:

– For X smooth and projective, the weight is pure

$$0 = W_{m-1}H^m(X,\mathbb{Q}) \subset W_m H^m(X,\mathbb{Q}) = H^m(X,\mathbb{Q}).$$

– If X is smooth (but possibly not complete), let $j\colon X \hookrightarrow \overline{X}$ be a smooth compactification of X, such that $D = \overline{X} - X$ is a normal crossing divisor. For all $m \geq 0$, then $W_{m-1}H^m(X,\mathbb{Q}) = 0$, the weights on $H^m(X,\mathbb{Q})$ are $\geq m$. Moreover

$$W_m H^m(X,\mathbb{Q}) = \operatorname{Im}\bigl(j^*\colon H^m(\overline{X},\mathbb{Q}) \to H^m(X,\mathbb{Q})\bigr).$$

In general, the weight filtration satisfies:

$$0 \subset W_m H^m(X,\mathbb{Q}) \subset \cdots \subset W_{2m}H^m(X,\mathbb{Q}) = H^m(X,\mathbb{Q}).$$

– For a singular and projective complex algebraic variety X of dimension d, consider its cubical hyperresolution $X_\bullet \to X$, where X_α is smooth, projective and $\dim(X_\alpha) \leq d - |\alpha| + 1$ [GNPP88]. This gives a weight spectral sequence:

$$E_1^{p,q}(X) = \bigoplus_{|\alpha|=p+1} H^q(X_\alpha,\mathbb{Q}) \Rightarrow H^{p+q}(X,\mathbb{Q}).$$

The weight filtration is given by a shift of the filtration induced on $H^*(X,\mathbb{Q})$ by the weight spectral sequence. For all $m \geq 0$ it satisfies

$$0 \subset W_0 H^m(X,\mathbb{Q}) \subset \cdots \subset W_m H^m(X,\mathbb{Q}) = H^m(X,\mathbb{Q}).$$

– Let X be a complex variety. Let $j\colon X \hookrightarrow \overline{X}$ be a compactification of X. Consider a hyperresolution of a pair $(X_\bullet, \overline{X}_\bullet) \to (X, \overline{X})$ such that $D_\alpha = \overline{X}_\alpha - X_\alpha$ is a normal crossing divisor for each α. Then

$$E_1^{-p,q}(X) = \bigoplus E_1^{-p-|\alpha|+1,q}(X_\alpha) \Rightarrow H^{q-p}(X,\mathbb{Q}).$$

Example 1.3.8 Let $\overline{C}$ be a connected compact Riemann surface of genus g, and $\Sigma = \{p_1, \ldots, p_m\} \subset \overline{C}$ a finite set of points. Consider the smooth, open surface $C := \overline{C} - \Sigma$. Then $H^1(C,\mathbb{Z})$ carries a natural $\mathbb{Z}$-MHS. The Hodge filtration on $H^1(C,\mathbb{Z}(1))$ is defined in terms of a filtered complex of holomorphic differentials on C with logarithmic poles along Σ [HodgeII, HodgeIII]. Poincaré duality gives us $H^1_\Sigma(\overline{C},\mathbb{Z}) \cong H_1(\Sigma,\mathbb{Z}) = 0$ and we have an exact sequence:

$$0 \to H^1(\overline{C},\mathbb{Z}) \to H^1(C,\mathbb{Z}) \to H^2_\Sigma(\overline{C},\mathbb{Z}) \cong \mathbb{Z}^{\oplus m} \to$$
$$\to H^2(\overline{C},\mathbb{Z}) \cong \mathbb{Z} \to H^2(C,\mathbb{Z}) = 0.$$

This sequence induces a short exact sequence of mixed Hodge structures:

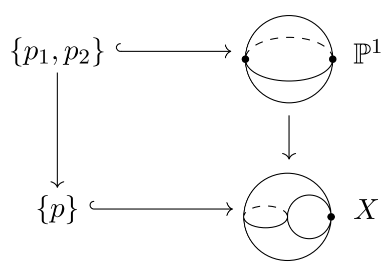
$$0 \to H^1(\overline{C},\mathbb{Z}) \to H^1(C,\mathbb{Z}) \to \mathrm{Ker}\big(H^2_\Sigma(\overline{C},\mathbb{Z}) \to H^2(\overline{C},\mathbb{Z})\big) \cong \mathbb{Z}^{m-1} \to 0.$$

The weights are given by: $W_2H^1(C,\mathbb{Z}) = H^1(C,\mathbb{Z})$,

$$W_1H^1(C,\mathbb{Z}) = \mathrm{Im}\big(H^1(\overline{C},\mathbb{Z}) \to H^1(C,\mathbb{Z})\big) = H^1(\overline{C},\mathbb{Z}) \cong \mathbb{Z}^{2g},$$

and $W_0H^1(C,\mathbb{Z}) = 0$. Then, the graded piece $\mathrm{Gr}^W_2 = H^1(C,\mathbb{Z}) = \mathbb{Z}^{m-1}$ with pure weight 2. Furthermore, $\mathrm{Gr}^W_1 = H^1(\overline{C},\mathbb{Z})$.

Example 1.3.9 Let $X = \{(x_0,x_1,x_2) \in \mathbb{P}^2_{\mathbb{C}} \mid x_0^3 + x_1^3 = x_0x_1x_3\}$ be the nodal cubic. This is a complex projective curve with $X_{\text{sing}} = \{[0:0:1]\} = p$ the singular locus. A cubical resolution of X is given by the diagram



Then we have an exact sequence on cohomology:

$$0 \to H^0(X,\mathbb{Z}) \to H^0(\mathbb{P}^1 \sqcup \{p\},\mathbb{Z}) \to H^0(\{p_1,p_2\},\mathbb{Z}) \to H^1(X,\mathbb{Z})$$
$$\to H^1(\mathbb{P}^1,\mathbb{Z}) \to H^1(\{p_1,p_2\},\mathbb{Z}) \to H^2(X,\mathbb{Z}) \to H^2(\mathbb{P}^1,\mathbb{Z}) \to 0.$$

Then $H^2(X,\mathbb{Z}) \xrightarrow{\cong} H^2(\mathbb{P}^1,\mathbb{Z})$, and the exact sequence becomes

$$0 \to \mathbb{Z} \to \mathbb{Z}^{\oplus 2} \to \mathbb{Z}^{\oplus 2} \to \mathbb{Z} \to 0 \to 0 \to \mathbb{Z} \to \mathbb{Z} \to 0.$$

All cohomologies are of pure weight except $H^1(X,\mathbb{Z})$. Taking the sequence induced by the graded piece Gr^W_0:

$$0 \to \mathbb{Z} \to \mathbb{Z}^{\oplus 2} \to \mathbb{Z}^{\oplus 2} \to W_0H^1(X,\mathbb{Z}) \to 0$$

we observe that $W_0H^1(X,\mathbb{Z})$ has rank 1, so $\mathrm{Gr}^W_0 H^1(X,\mathbb{Z}) = \mathbb{Z}$. By the exact sequence that induces Gr^W_1, we can see that $W_1H^1(X,\mathbb{Z}) = 0$ and $\mathrm{Gr}^W_1 H^1(X,\mathbb{Z}) = 0$.

Extensions of mixed Hodge structures

The category **MHS** does not have enough injectives, but there is a general theory due to Verdier-Yoneda for extensions of mixed Hodge structures. In any abelian category $\mathcal{A}$, with or without enough injectives/projectives, the following formula provides a definition for the Ext-groups in general [PS08, Appendix A.2.6]:

$$\mathrm{Ext}^n(A,B) := \mathrm{Hom}_{\mathbf{D}^+(\mathcal{A})}(A, B[n]),$$

in terms of Verdier's derived category of $\mathcal{A}$. This definition extends the classical notion of Ext in terms of injective resolutions.

1.3.10 Yoneda extensions. Let $A, B \in \mathcal{A}$, and $n \in \mathbb{Z}_{>0}$. An **extension of degree** n of A by B is an exact sequence

$$E: \ [\ 0 \to B \to K^{-n} \to \cdots \to K^{-1} \to A \to 0\ ].$$

Given two extensions E and E' of the same degree, we say are **congruent** if there is a commutative diagram

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & B & \longrightarrow & K^{-n} & \longrightarrow & \cdots & \longrightarrow & K^{-1} & \longrightarrow & A & \longrightarrow & 0 \\ & & \| & & \downarrow & & & & \downarrow & & \| & & \\ 0 & \longrightarrow & B & \longrightarrow & K'^{-n} & \longrightarrow & \cdots & \longrightarrow & K'^{-1} & \longrightarrow & A & \longrightarrow & 0. \end{array}$$

With the Baer sum the equivalence class sets $\mathrm{Ext}_{\mathrm{Yon}}(A,B)$ form a group [PS08, A.2.6].

Proposition 1.3.11 [PS08, Lemma A.32] *Let $\mathcal{A}$ be an abelian category. For $A, B \in \mathcal{A}$, there are functorial isomorphisms*

$$\mathrm{Ext}^n_{\mathrm{Yon}}(A,B) \xrightarrow{\cong} \mathrm{Ext}^n(A,B) = \mathrm{Hom}_{\mathbf{D}^+(\mathcal{A})}(A, B[n]).$$

An important result about Ext-groups is the following:

Lemma 1.3.12 [PS08, Lemma A.33] *If $\mathrm{Ext}^k(A,-)$ is right exact for all $A \in \mathcal{A}$, then $\mathrm{Ext}^n(A,B) = 0$ for $n \geq k+1$ and all $A, B \in \mathcal{A}$.*

1.3.13 Extensions of MHS. The category of mixed Hodge structures is abelian, hence Yoneda Ext-functor gives a form to consider the groups $\mathrm{Ext}^n_{\mathrm{MHS}}(A,B)$. This is defined for $n \geq 1$, for $n = 0$ we put $\mathrm{Ext}^0_{\mathrm{MHS}} = \mathrm{Hom}_{\mathrm{MHS}}$.

Carlson's description. In the category of mixed Hodge structures, one can explicitly describe the particular case $\mathrm{Ext}^1(A,B)$, using *extensions* [Car80].

Definition 1.3.14 Let A and B be two mixed Hodge structures.

i) An **extension of MHS** is an exact sequence of mixed Hodge structures
$$0 \longrightarrow A \xrightarrow{i} H \xrightarrow{\pi} B \longrightarrow 0.$$
We say that H is an extension of B by A. A **section** is a morphism $s\colon B \to H$, such that $\pi \circ s = 1_B$; an extension with section is **split**.

ii) A **morphism of extensions** is a commutative diagram
$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"] & H \arrow[r] \arrow[d] & B \arrow[r] \arrow[d, "\beta"] & 0 \\ 0 \arrow[r] & A' \arrow[r] & H' \arrow[r] & B' \arrow[r] & 0. \end{tikzcd}$$
A **congruence** of extensions is an isomorphism for which α and β are each the identity.

Remark. Split extensions are congruent to $0 \to A \to A \oplus B \to B \to 0$.

Definition 1.3.15 The set of isomorphisms classes of extensions of B by A, is denoted by
$$\mathrm{Ext}^1_{\mathbf{MHS}}(B,A) = \frac{\text{extensions}}{\text{congruence}}.$$
The group structure is imposed by the Baer summation [PS08, A.30].

Definition 1.3.16 Let H be a mixed Hodge structure. We define
$$J(H) := \frac{W_0 H_{\mathbb{C}}}{F^0 W_0 H_{\mathbb{C}} + W_0 H_{\mathbb{Q}}}.$$
$$J^p(H) := \frac{W_{2p} H_{\mathbb{C}}}{F^p W_{2p} H_{\mathbb{C}} + W_{2p} H_{\mathbb{Q}}}.$$
The **generalized Jacobian** of H.

Given two filtered vector spaces $(A, W_\bullet)$ and $(B, W_\bullet)$ with increasing filtrations, then $\mathrm{Hom}(A,B)$ has an induced filtration given by
$$W_n \mathrm{Hom}(A,B) = \{\phi\colon A \to B \mid \phi(W_k A) \subset W_{k+n} B\}.$$
Similarly for decreasing filtrations $F^\bullet$. The following result on extension of mixed Hodge structures was proved by Carlson [Car80].

Theorem 1.3.17 (Carlson) *Let A and B be a mixed Hodge structure. Then, there exists the following identification:*

$$\operatorname{Ext}^1_{\mathrm{MHS}}(A,B) \cong \frac{W_0\operatorname{Hom}(A,B)_{\mathbb{C}}}{W_0 \cap F^0\operatorname{Hom}(A,B)_{\mathbb{C}} + W_0\operatorname{Hom}(A,B)}.$$

Where $\operatorname{Hom}(A,B)$ *is viewed as a mixed Hodge structure.*

Example 1.3.18 For $m < n$, the group $\operatorname{Ext}^1\big(\mathbb{Z}(m),\mathbb{Z}(n)\big)$ is

$$\begin{aligned}\operatorname{Ext}^1\big(\mathbb{Z}(m),\mathbb{Z}(n)\big) &\cong J\big(\operatorname{Hom}(\mathbb{Z}(m),\mathbb{Z}(n))\big)\\ &\cong J\big(\mathbb{Z}(n-m)\big) \cong \mathbb{C}/(2\pi i)^{n-m}\mathbb{Z} = \mathbb{C}^*.\end{aligned}$$

Example 1.3.19 For H any mixed Hodge structure

$$\operatorname{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0),H) \cong \frac{W_0H_{\mathbb{C}}}{F^0W_0H_{\mathbb{C}} + W_0H_{\mathbb{Q}}}.$$

Example 1.3.20 Let C be a smooth, projective complex curve. $H^1(C,\mathbb{Z})$ is a pure Hodge structure of weight 1, and there is a natural isomorphism

$$\operatorname{Ext}^1_{\mathrm{MHS}}\big(\mathbb{Z}(-1),H^1(C,\mathbb{Z})\big) = \frac{H^1(C,\mathbb{C})}{H^1(C,\mathbb{Z}) \oplus F^1H^1(C,\mathbb{C})} \cong \frac{H^{0,1}(C)}{H^1(C,\mathbb{Z})}.$$

This is an abelian variety, called the **Jacobian variety** of the curve C, that identifies via the exponential map with $\operatorname{Ker}[H^1(C,\mathcal{O}_C^*) \to H^2(C,\mathbb{Z})]$. For a finite number of points $\Sigma = \{p_1,\ldots,p_m\} \subset C$ with $\sum p_i = 0$, the choice of mixed Hodge estructure on $H^1(C-\Sigma,\mathbb{Z})$ defines an **Abel-Jacobi morphism** on the curve C in terms of classes of extensions of mixed Hodge structures. In general, if X is a smooth and projective variety, then

$$\operatorname{Ext}^1_{\mathrm{MHS}}\big(\mathbb{Z}(-q),H^{2q-1}(X,\mathbb{Z})\big) \cong \frac{H^{2q-1}(X,\mathbb{C})}{F^qH^{2q-1}(X,\mathbb{C}) + H^{2q-1}(X,\mathbb{Z})}.$$

The description of the Abel-Jacobi morphism is given in the next chapter.

By Carlson's description of $\operatorname{Ext}^1_{\mathrm{MHS}}$, we have that the category of mixed Hodge structures has **cohomological dimension** one, i.e. all higher extensions vanish.

Theorem 1.3.21 *For any two mixed Hodge structures A, B, and $j \geq 2$. The functor* $\operatorname{Ext}^1_{\mathrm{MHS}}(A,-)$ *is right exact, and then* $\operatorname{Ext}^j_{\mathrm{MHS}}(A,B) = 0$.

Proof. This is consequence of Lemma 1.3.12, and the fact that if $H \to H'$ is a surjective morphism of mixed Hodge structures, then

$$\operatorname{Ext}^1_{\mathrm{MHS}}(A,H) \to \operatorname{Ext}^1_{\mathrm{MHS}}(A,H')$$

is surjective. $\square$

1.4. Deligne-Beilinson (co)homology

We review the construction of Deligne-Beilinson cohomology and homology. For further details, see [Jan88] and [EV88]. Let $\mathbb{A} \subset \mathbb{R}$ be a subring and $p \geq 0$. We define the **Tate twist** $\mathbb{A}(p) := (2\pi i)^p \mathbb{A}$.

Definition 1.4.1 Let X be a smooth projective complex variety of dimension d. The p'th **Deligne complex** $\mathbb{A}_{\mathcal{D}}^{\bullet}(p)$ is the complex

$$\mathbb{A}_{\mathcal{D}}^{\bullet}(p) : 0 \to \mathbb{A}(p) \to \underbrace{\mathcal{O}_X \to \Omega_X^1 \to \cdots \to \Omega_X^{p-1}}_{=:\Omega_X^{\bullet<p}} \to 0 \to \cdots$$

in degrees 0 up to p. In this complex, $\Omega_X^{\bullet}$ denotes the complex of holomorphic differential forms on $X(\mathbb{C})$. The q'th **Deligne cohomology** of X with coefficients in $\mathbb{A}(p)$ is given by the hypercohomology

$$H_{\mathcal{D}}^q(X, \mathbb{A}(p)) := \mathbb{H}_{\text{an}}^q(X, \mathbb{A}_{\mathcal{D}}(p)).$$

Deligne cohomology via a cone complex. Consider again the complex of holomorphic differential forms $\Omega_X^{\bullet}$, and the two subcomplexes

$$\mathbb{A}(p)^{\bullet} : 0 \to \mathbb{A}(p) \to 0 \to \cdots$$

$$\Omega_X^{\bullet \geq p} : 0 \to 0 \to \cdots \to 0 \to \Omega_X^p \to \Omega_X^p \to \cdots$$

with natural morphisms $\epsilon\colon \mathbb{A}(p) \subset \mathbb{C} \to \Omega_X^{\bullet}$ and $\iota\colon \Omega_X^{\bullet \geq p} \to \Omega_X^{\bullet}$. Then, the cone complex

$$\mathrm{Cone}^{\bullet}\big(\mathbb{A}(p) \oplus \Omega_X^{\bullet \geq p} \xrightarrow{\epsilon - \iota} \Omega_X^{\bullet}\big)[-1]$$

is given by:

$$\mathbb{A}(p) \to \mathcal{O}_X \xrightarrow{d} \Omega_X \xrightarrow{d} \cdots \xrightarrow{d} \Omega_X^{p-2} \xrightarrow{(0,d)} (\Omega_X^p \oplus \Omega_X^{p-1})$$

$$\xrightarrow{\delta} (\Omega_X^{p+1} \oplus \Omega_X^p) \xrightarrow{\delta} \cdots \xrightarrow{\delta} (\Omega_X^d \oplus \Omega_X^{d-1}) \to \Omega^d.$$

There is a natural inclusion of the Deligne complex into this complex

$$i\colon \mathbb{A}_{\mathcal{D}}^{\bullet}(p) \to \mathrm{Cone}^{\bullet}\big(\mathbb{A}(p) \oplus \Omega_X^{\bullet \geq p} \to \Omega_X^{\bullet}\big)[-1].$$

By the holomorphic Poincaré lemma, this morphism is a quasi-isomorphism. Thus, we have

$$H_{\mathcal{D}}^q(X, \mathbb{A}(p)) \cong \mathbb{H}^q\big(X, \mathrm{Cone}(\mathbb{A}(p) \oplus \Omega_X^{\bullet \geq p} \to \Omega_X^{\bullet})[-1]\big).$$

Definition 1.4.2 The product structure on Deligne cohomology comes from the product on the level of complexes:

$$\cup\colon \mathbb{A}_{\mathcal{D}}(p) \otimes \mathbb{A}_{\mathcal{D}}(q) \to \mathbb{A}_{\mathcal{D}}(p+q)$$

given by

$$x \cup y = \begin{cases} x \cdot y, & \text{if } \deg x = 0 \\ x \wedge dy, & \text{if } \deg x > 0 \text{ and } \deg y = q > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Example 1.4.3 Take $\mathbb{A} = \mathbb{Z}$. For $p = 0$, Deligne cohomology is the singular cohomology $H^i_{\mathcal{D}}(X, \mathbb{A}(0)) = H^i(X, \mathbb{A})$. For $p = 1$, we have a quasi-isomorphism

$$\exp\colon \mathbb{Z}_{\mathcal{D}}(1) \to \mathcal{O}^*_X[-1]$$

via the exponential map. Explicitly, the quasi-isomorphism is given by the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (2\pi i)\mathbb{Z} & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \exp} & & \downarrow{\scriptstyle \exp} & & \downarrow & & \\ 1 & \longrightarrow & 1 & \longrightarrow & \mathcal{O}^*_X & \longrightarrow & 1 & \longrightarrow & \cdots \end{array}$$

Hence, $H^2_{\mathcal{D}}(X, \mathbb{Z}(1)) \cong H^1(X, \mathcal{O}^*_X) = \mathrm{Pic}(X)$. Moreover, we have an exact sequence:

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \to \mathrm{NS}(X) \to 0,$$

where $\mathrm{NS}(X)$ is the Neron-Severi group, and $\mathrm{Pic}^0(X) = \mathrm{Jac}^1(X)$. In general, the complex $\mathbb{Z}_{\mathcal{D}}(p)$ fits into the short exact sequence

$$0 \to \Omega^{\bullet<p}_X[-1] \to \mathbb{Z}_{\mathcal{D}}(p) \to \mathbb{Z}(p) \to 0$$

and leads, by the Hodge theory, to the following short exact sequence

$$0 \longrightarrow \frac{H^{i-1}(X, \mathbb{C})}{F^q H^{i-1}(X, \mathbb{C}) + H^{i-1}(X, \mathbb{Z}(q))} \longrightarrow H^i_{\mathcal{D}}(X, \mathbb{Z}(q))$$

$$\longrightarrow F^q H^i(X, \mathbb{C}) \bigcap H^i(X, \mathbb{Z}(q)) \longrightarrow 0.$$

Now, using the interpretation of the Ext-groups for mixed Hodge structures, we have the exact sequence:

$$0 \longrightarrow \mathrm{Ext}_{\mathrm{MHS}}\big(\mathbb{Z}, H^{i-1}(X, \mathbb{Z}(q)\big) \longrightarrow H^i_{\mathcal{D}}\big(X, \mathbb{Z}(q)\big)$$
$$\longrightarrow \mathrm{Hom}_{\mathrm{MHS}}\big(\mathbb{Z}, H^i(X, \mathbb{Z}(q)\big) \longrightarrow 0.$$

In particular, the case when $i = 2q$ gives us an exact sequence

$$0 \to \mathrm{Jac}^q(X) \to H^{2q}_{\mathcal{D}}(X, \mathbb{Z}(q)) \to \mathrm{Hdg}^q(X) \to 0.$$

Deligne cycle-class morphism.

Let X be a smooth, projective variety over $\mathbb{C}$. Denote by $\mathcal{Z}^q(X)$ the group of algebraic cycles on X of codimension q.

Theorem 1.4.4 *There exists a morphism*

$$\mathrm{cl}_{\mathcal{D}} \colon \mathcal{Z}^q(X) \to H^{2q}_{\mathcal{D}}\big(X, \mathbb{Z}(q)\big).$$

Moreover, this morphism factor through $\mathrm{CH}^q(X)$*, and we have the following commutative diagram:*

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{CH}^q_{\mathrm{hom}}(X) \arrow[r] \arrow[d, "\mathrm{AJ}"'] & \mathrm{CH}^q(X) \arrow[d, "\mathrm{cl}_{\mathcal{D}}"'] \arrow[dr] & & \\
0 \arrow[r] & \mathrm{Jac}^q(X) \arrow[r] & H^{2q}_{\mathcal{D}}\big(X, \mathbb{Z}(q)\big) \arrow[r] & \mathrm{Hdg}^q(X) \arrow[r] & 0
\end{tikzcd}$$

where $\mathrm{CH}^q_{\mathrm{hom}}(X) \subset \mathrm{CH}^q(X)$ *are the cycles classes homologically equivalent to zero.*

A brief description of the cycle-class morphism and the Abel-Jacobi morphism in the case of classical Chow groups, will be given in the next chapter (Section 2.1). The Deligne cohomology admits an extension to an arbitrary (possibly non-compact or singular) algebraic variety X over $\mathbb{C}$.

1.4.5 Let X be a smooth, quasi-projective complex algebraic variety. A **compactification** for X is a birational inclusion $j \colon X \hookrightarrow \overline{X}$ of X as a Zariski open of a smooth projective variety $\overline{X}$ [Nag62], such that the complement $D := \overline{X} \setminus X$ is a normal crossing divisor [Hir64]. Denote by $\Omega^\bullet_{\overline{X}}(\log D)$ the complex of meromorphic forms on $\overline{X}$, holomorphic on X, with at most logarithmic poles along D.

Definition 1.4.6 The **Deligne-Beilinson complex** is given by

$$\mathbb{A}^{\bullet}_{\mathcal{DB}}(p) := \mathrm{Cone}\big(Rj_*\mathbb{A}(p) \oplus F^p\Omega^{\bullet}_{\overline{X}}(\log D) \to Rj_*\Omega^{\bullet}_X\big)[-1].$$

The **Deligne-Beilinson cohomology** of X with coefficients in $\mathbb{A}(p)$ is defined as anaytic hypercohomology

$$H^i_{\mathcal{DB}}(X, \mathbb{Z}(p)) := \mathbb{H}^i_{\mathrm{an}}(\overline{X}, \mathbb{A}^{\bullet}_{\mathcal{DB}}(p)).$$

We can show that this is independent of the compactification [EV88, Lemma 2.8]. The Deligne-Beilinson cohomology can be extended to the case of a smooth simplicial scheme $X_\bullet$, by considering the corresponding complex over each component X_p. For a singular variety X, a semi-simplicial hyperresolution $X_\bullet \to X$ is a semi-simplicial scheme $X_\bullet$ with projective and smooth components X_p, together with a morphism to X which satisfies cohomological descent [GNPP88]. The Deligne cohomology is given by

$$H^q_{\mathcal{D}}(X, \mathbb{A}(p)) := \mathbb{H}^q_{\mathrm{an}}(X_\bullet, \mathbb{A}_{\mathcal{D}}(p))$$

where $\mathbb{A}_{\mathcal{D}}(p)$ is considered as a complex of analytic sheaves on $X_\bullet$. This definition is independent of choice of hyperresolution [Jan88]. For a non-compact case, we use a "simplicial compactification" $X_\bullet \hookrightarrow \overline{X}_\bullet$, where each $X_p \hookrightarrow \overline{X}_p$ is a good compactification, such that $D_p = \overline{X}_p \setminus X_p$ is again a normal crossing divisor. We may compute the Deligne-Beilinson complex $\mathbb{A}_{\mathcal{DB}}(p)$ on $\overline{X}_\alpha$, and we can arrange these complexes to be functorial with respect to the face maps of $\overline{X}_\bullet$. Then the Deligne-Beilinson complexes on the $\overline{X}_p$ organize into a simplicial complex over the simplicial scheme $\overline{X}_\bullet$. Taking cohomology, we obtain the Deligne-Beilinson cohomology

$$H^*_{\mathcal{DB}}(X, \mathbb{A}(r)) := \mathbb{H}^*(X_\bullet, \mathbb{A}^{\bullet}_{\mathcal{DB}}(r)).$$

This is independent of the choice of the compactification [Jan88].

1.4.7 Deligne homology. In order to have Poincaré duality, the Deligne homology is constructed, this is a counterpart of Deligne-Beilinson cohomology. The construction is based on currents and C^∞-chains. For the definition, we need to introduce some notation [Jan88]:

i) Let $\Omega^{p,q}_{X^\infty}$ be the sheaf of (p,q)-forms on X. Denote by $\mathcal{D}^{p,q}_{X^\infty}$ the sheaf of currents acting on $\Omega^{-p,-q}_{X^\infty}$. Thus for an open set $U \subseteq X$, we have

$$\mathcal{D}^{p,q}_{X^\infty}(U) := \{\text{continuous linear functionals on } \Gamma_c(U, \Omega^{-p,-q}_{X^\infty})\}.$$

ii) Sheaves $\mathcal{D}^{\bullet,\bullet}_{X^\infty}$ and $\Omega^{\bullet,\bullet}_{X^\infty}$ naturally form double complexes. Denote by $\mathcal{D}^{\bullet}_{X^\infty}$ and $\Omega^{\bullet}_{X^\infty}$ the corresponding total complexes. We have two Hodge filtrations

$$F^i\Omega^n_{X^\infty} = \bigoplus_{p+q=n, p\geq i} \Omega^{p,q}_{X^\infty}, \qquad F^i\mathcal{D}^n_{X^\infty} = \bigoplus_{p+q=n, p\geq i} \mathcal{D}^{p,q}_{X^\infty}.$$

iii) Consider $\big(C_\bullet(X, \mathbb{A}(r)), d\big)$ the complex of singular C^∞-chains on X with coefficients in $\mathbb{A}(r)$. Define the change of index by $C_X^{2d-\bullet} := C_{\bullet,X}$. There is a morphism

$$\epsilon\colon C^\bullet\big(X, \mathbb{A}(r)\big) \to \mathcal{D}^\bullet_{X^\infty}$$

given by integrations over chains.

Definition 1.4.8 The **Deligne homology** $H_*^{\mathcal{D}}(X, \mathbb{A}(r))$ is given by the cohomology of the complex

$$\mathrm{Cone}\big(C^\bullet(X, \mathbb{A}(r)) \oplus F^r\mathcal{D}^\bullet_X(X) \xrightarrow{\epsilon - l} \mathcal{D}^\bullet_X(X)\big)[-1].$$

Deligne homology admits an extension to smooth quasi-projective varieties, by considering compactifications and logarithmic singularities. For singular varieties, X can be replaced by a simplicial variety with smooth components $X_\bullet$. Then we can take a good simplicial compactification $\overline{X}_\bullet$ and $Y_\bullet = \overline{X}_\bullet - X_\bullet$ a simplicial normal crossing divisor, so that technique of logarithmic singularities applies [Jan88].

1.5. Appendix: Hypercohomology

The notion of hypercohomology is widely used throughout this work, so in this small appendix we will give its definition. Hypercohomology is a generalization of sheaf cohomology. Among other things, the definition of Deligne-Beilinson cohomology, absolute Hodge cohomology, motivic cohomology are given in terms of hypercohomology.

1.5.1. Let X be a topological space. Denote by $\mathbf{Sh}(X)$ the abelian category of sheaves of abelian groups on X, and by $\Gamma(X, -)\colon \mathbf{Sh}(X) \to \mathbf{Ab}$ the global section functor. The category $\mathbf{Sh}(X)$ has enough injectives.

Definition 1.5.1 Let $\mathcal{F}^\bullet$ be a bounded below complex of sheaves on X. For $i \geq 0$, we define the **hypercohomology** of the complex $\mathcal{F}^\bullet$ as

$$\mathbb{H}^i(X, \mathcal{F}^\bullet) := R^i\Gamma(X, \mathcal{F}^\bullet) = H^i\big(\mathrm{Tot}(\Gamma(X, \mathcal{I}^{\bullet,\bullet}))\big),$$

where $\mathcal{F}^\bullet \to \mathcal{I}^{\bullet,\bullet}$ is an injective resolution. This construction defines a functor

$$\mathbb{H}^i(X,-)\colon \mathbf{C}^+(\mathbf{Sh}(X)) \to \mathbf{Ab}.$$

We can also use this for a single sheaf $\mathcal{F}$ viewed as a complex in degree 0, and we write $H^i(X,\mathcal{F})$ instead of $\mathbb{H}^i(X,\mathcal{F})$. In fact, we have a **hypercomology spectral sequence**

$$E_2^{p,q} = H^p(X,\mathcal{H}(\mathcal{F}^\bullet)) \Rightarrow \mathbb{H}^{p+q}(X,\mathcal{F}^\bullet).$$

Corollary 1.5.2 *If $f\colon A_1^\bullet \to A_2^\bullet$ is a quasi-isomorphism, then*

$$F(f)\colon \mathbb{H}^i(F(A_1^\bullet)) \to \mathbb{H}^i(F(A_2^\bullet))$$

is an isomorphism on hypercohomology groups.

Chapter 2

The Abel-Jacobi morphism for Bloch's higher Chow groups

The main purpose of this chapter is to collect the principal tools to establish the KLM-formula [KLM06, KL07]. For this, we start with the classical case of Riemann surfaces, where the Abel-Jacobi morphism is given in terms of integrals to the Jacobian. In varieties of higher dimension, this is a morphism from cycles homologous to zero to Griffiths's intermediate Jacobian.

2.1. The classical case

Let X be a smooth, projective complex curve of genus g. Then, $H^1(X,\mathbb{Z})$ defines a pure Hodge structure of weight 1, so $H^1(X,\mathbb{C}) = H^{0,1}(X) \oplus H^{1,0}(X)$ with $F^0 = H^1(X,\mathbb{C}) \supset F^1 = H^{1,0}(X)$. The **Jacobian** of X is given by

$$\mathrm{Jac}(X) := \frac{H^1(X,\mathbb{C})}{F^1H^1(X,\mathbb{C}) + H^1(X,\mathbb{Z})} \cong \frac{H^0(X,\Omega_X)^\vee}{H_1(X,\mathbb{Z})},$$

wich is an abelian variety. Let $\alpha = \sum_{i=1}^l n_i\alpha_i \in \mathrm{CH}^1(X)$ be a non-trivial divisor with degree $\sum_i n_i = 0$. The classical way to compute the **Abel-Jacobi morphism** is: Choose a continuous chain γ such that $\partial(\gamma) = \alpha$ and a basis $\omega_1, \ldots, \omega_g$ of holomorphic 1-forms on X. Then the vector of **periods**

$$\Big(\int_\gamma \omega_1, \ldots, \int_\gamma \omega_g\Big)$$

defines the **Abel-Jacobi class**. In this case, the Abel-Jacobi morphism

$$\mathrm{AJ}^1\colon \mathrm{CH}^1(X)_{\deg\ 0} \to \mathrm{Jac}(X)$$

gives an isomorphism. Via Carlson's extensions [Car80], we have another aproximation. Consider the exact cohomology sequence with support on

$|\alpha|$. Observe that $H^1_{|\alpha|}(X,\mathbb{Z}) \cong H_1(|\alpha|,\mathbb{Z}) = 0$, then we have a sequence of mixed Hodge structures (**MHS**):

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & H^1(X) \arrow[r] & H^1(X-|\alpha|) \arrow[r] & H^2_{|\alpha|}(X) \arrow[r] \arrow[d, "\cong"] & H^2(X) \arrow[d, "\cong"] \\
H^1_{|\alpha|}(X) \arrow[ur] & & & \bigoplus_i \mathbb{Z}(-1) \arrow[r, "\Sigma"] & \mathbb{Z}(-1).
\end{tikzcd}$$

The cycle α defines a non-zero class in $H^2_{|\alpha|}(X,\mathbb{Z})$ mapping to zero in $H^2(X,\mathbb{Z})$. Via pullback, we obtain another short exact sequence of **MHS**:

$$0 \longrightarrow H^1(X) \longrightarrow E \longrightarrow \mathbb{Z}(-1) \longrightarrow 0.$$

Then, the Abel-Jacobi class of α is an extension class of this sequence in the category **MHS**, $\{E\} \in \mathrm{Ext}^1_{\mathrm{MHS}}\big(\mathbb{Z}(-1), H^1(X,\mathbb{Z})\big)$. The construction of Abel-Jacobi for curves can be generalized to any smooth, projective variety X over $\mathbb{C}$, and $\alpha \in \mathrm{CH}^q(X)$ an algebraic cycle which is homologous to zero. Then $\alpha = \partial(\gamma)$, for some γ a $2d-2q+1$ real dimensional chain in X. The general Abel-Jacobi morphism is

$$\mathrm{AJ}^q\colon \mathrm{CH}^q(X)_{\mathrm{hom}} \to \mathrm{Jac}^q(X)$$

in **Griffiths' intermediate Jacobian**, wich is a complex torus defined as

$$\begin{aligned}\mathrm{Jac}^q(X) = J\big(H^{2q-1}(X)\big) \quad &:= \quad \frac{H^{2q-1}(X,\mathbb{C})}{F^qH^{2q-1}+H^{2q-1}(X,\mathbb{Z})} \\ &\cong \quad \frac{F^{d-q+1}H^{2d-2q+1}(X)^\vee}{H_{2d-2q+1}(X,\mathbb{Z})}.\end{aligned}$$

Let $\{w\} \in F^{d-q+1}H^{2d-2q+1}(X,\mathbb{C})$. The Abel-Jacobi class is given by

$$\mathrm{AJ}^q(\alpha)(\{w\}) = \frac{1}{(2\pi i)^{d-q}} \int_\gamma \omega \ / \ \{\text{periods}\}.$$

On the other hand, as in the example of curves, the cycle α defines an extension of **MHS**:

$$0 \longrightarrow H^{2q-1}(X) \longrightarrow E \longrightarrow \mathbb{Z}(-q) \longrightarrow 0$$

and we have a class $\{E\} \in \mathrm{Ext}^1_{\mathrm{MHS}}\big(\mathbb{Z}(-q), H^{2q-1}(X,\mathbb{Z})\big) \cong \mathrm{Jac}^q(X)$, see [Car80]. Hence a reformulated Abel-Jacobi morphism

$$\mathrm{AJ}^q\colon \mathrm{CH}^q(X)_{\mathrm{hom}} \to \mathrm{Ext}^1_{\mathrm{MHS}}\big(\mathbb{Z}(-q), H^{2q-1}(X,\mathbb{Z})\big).$$

This result can be generelised to *Bloch's higher Chow groups* [Blo86a], wich is motivic cohomology in the smooth case (2.2.9), but a Borel-Moore homology theory in the general case.

2.2. Bloch's higher Chow groups

Cycle complexes. We consider equidimensional and reduced quasi-projective varieties over a field k. In this work, we define the cubical version of the theory of higher Chow groups [Blo86a], [Lev94]. Let k be a field. We consider the **algebraic cube** $\square^1 := \mathbb{P}^1_k - \{1\}$ and $\square^n = (\square^1)^n$ with coordinates $(t_1, \ldots, t_n)$. Codimension one faces on $\square^n$ are obtained by setting $t_i = 0, \infty$. Intersecting these faces gives us higher codimension faces. Let $C^r(X, n)$ be the free abelian group generated by subvarieties of $X \times \square^n$ of codimension r and meeting all the faces of the cubes properly. There is a map:

$$\partial_n := \sum_{i=1}^{n} (-1)^i (\partial_i^\infty - \partial_i^0),$$

where ∂_i^∞ (respectively ∂_i^0) is the pull-back to the face $t_i = \infty$ (respectively $t_i = 0$). The map ∂_n satisfies $\partial_{n-1} \circ \partial_n = 0$, effectively making $\big(C^r(X, n), \partial_n\big)$ a complex. Let $D^r(X, n)$ be the subgroup of $C^r(X, n)$ generated by cycles which are the pull-back of some cycle on $X \times \square^{n-1}$ via a projection of the form $\square^n \to \square^{n-1}$: $(t_1, \ldots, t_n) \mapsto (t_1, \ldots, \hat{t_i}, \ldots, t_n)$. We call such cycles **degenerate cycles**. Then $\{D^q(X, n), \partial_n\}$ is a subcomplex of $\{(C^r(X, n), \partial_n\}$. Taking the quotient $\mathcal{Z}^r(X, \bullet) := C^r(X, \bullet)/D^r(X, \bullet)$, we have the complex

$$\mathcal{Z}^r(X, \bullet) : \cdots \longrightarrow \mathcal{Z}^r(X, n+1) \longrightarrow \mathcal{Z}^r(X, n) \longrightarrow \mathcal{Z}^r(X, n-1) \longrightarrow \cdots$$

called **Bloch's cycle complex** of X in codimension r.

Definition 2.2.1 The **higher Chow groups** are the homology groups of the cycle complex:

$$\mathrm{CH}^r(X, n) := H_n\big(\mathcal{Z}^r(X, \bullet)\big).$$

Note that $\mathrm{CH}^r(X, 0) = \mathrm{CH}^r(X)$, the classical Chow group of codimension r cycles on X modulo rational equivalence [Ful84]

$$\cdots \xrightarrow{\partial} \mathcal{Z}^r(X, 2) \xrightarrow{\partial} \mathcal{Z}^r(X, 1) \xrightarrow{\partial} \mathcal{Z}^r(X) \longrightarrow 0$$

the image of $\partial = \partial_1^0 - \partial_1^\infty$ is exactly the cycles rationally equivalent to zero, and $\mathrm{CH}^r(X)$ is the cokernel. For an equi-dimensional variety X we set $\mathcal{Z}_s(X, \bullet) = \mathcal{Z}^{\dim X - s}(X, \bullet)$. For codimension reasons, these groups vanish: $\mathrm{CH}^r(X, n) = 0$ for $r > n + \dim(X)$.

2.2.2 Formal properties. We give a list of fundamental properties of $\mathrm{CH}^*(X, -)$, see [Blo86a], [Blo86b] and [Lev94].

– **Funtoriality.** Let $f\colon X \to Y$ be a morphism of k-varieties. If f is proper, we have an induced push-forward morphism of cycle complexes $f_*\colon \mathcal{Z}_r(X, \bullet) \to \mathcal{Z}_r(Y, \bullet)$. If f is flat, the pull-back (contravariant) of cycle gives the morphism of complexes $f^*\colon \mathcal{Z}^r(Y, \bullet) \to \mathcal{Z}^r(X, \bullet)$.

– **Products.** If X is smooth, quasi-projective and equidimensional, we have the "internal" intersection product

$$\mathrm{CH}^p(X, m) \times \mathrm{CH}^q(X, n) \to \mathrm{CH}^{p+q}(X, m+n).$$

– **Homotopy invariance.** If X is equidimensional, we have

$$\mathrm{CH}^*(X, n) \cong \mathrm{CH}^*(X \times \mathbb{A}^1, n).$$

– **Projection formula.** Let $f\colon X \to Y$ be a proper morphism of smooth schemes over k, $\alpha \in \mathrm{CH}^*(X, \cdot)$ and $\beta \in \mathrm{CH}^*(Y, \cdot)$. Then

$$f_*\big(\alpha \cdot f^*(\beta)\big) = f_*(\alpha) \cdot \beta.$$

– **Projective bundle formula.** Let X be a smooth and quasi-projective variety. If $\mathcal{E}$ is a vector bundle of rank $n+1$ over X with $\mathbb{P}(\mathcal{E}) \to X$ its projectivization, then $\mathrm{CH}^*(\mathbb{P}(\mathcal{E}), -)$ is a free $\mathrm{CH}^*(X, -)$-module with generators $1, \xi, \ldots, \xi^n$, where $\xi = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \in \mathrm{CH}^1(\mathbb{P}(\mathcal{E}))$.

– **Localization.** If $Z \subset X$ a closed subvariety of pure codimension d, with X quasi-projective, then the diagram of complex with natural maps

$$\mathcal{Z}^{r-d}(Z, \cdot) \longrightarrow \mathcal{Z}^r(X, \cdot) \longrightarrow \mathcal{Z}^r(X \setminus Z, \cdot)$$

can be extended to a distinguished triangle in the derived category, giving rise to the long exact **localization sequence**:

$$\to \mathrm{CH}^{r-d}(Z, n) \to \mathrm{CH}^r(X, n) \to \mathrm{CH}^r(X \setminus Z, n) \to \mathrm{CH}^{r-d}(Z, n-1) \to .$$

The localization property of cycle complexes gives us a blow-up formula

Theorem 2.2.3 *Consider the Cartesian square*

$$\begin{tikzcd} W \arrow[r] \arrow[d] & V \arrow[d, "p"] \\ U \arrow[r, "i"] & X \end{tikzcd}$$

where p is proper, i is a closed inmersion, and p induces an isomorphism $(V-W) \xrightarrow{\cong} (X-U)$. Then there is a distinguished triangle of the form

$$\mathcal{Z}_r(W,\bullet) \longrightarrow \mathcal{Z}_r(V,\bullet) \oplus \mathcal{Z}_r(U,\bullet) \longrightarrow \mathcal{Z}_r(X,\bullet) \xrightarrow{[1]}$$

in the derived category of abelian groups.

Example 2.2.4 For X smooth and irreducible, in codimension one S. Bloch [Blo86a] shows that:

$$\mathrm{CH}^1(X,n) = \begin{cases} \mathrm{Pic}(X), & n=0 \\ \Gamma(X,\mathcal{O}_X^*), & n=1 \\ 0, & n>1. \end{cases}$$

Example 2.2.5 If $X = \mathrm{Spec}(k)$ for a field k, by the work of Nesterenko-Suslin [NS89] and Totaro [Tot92], $\mathrm{CH}^n(k,n) \cong K_n^M(k)$ are the Milnor K-groups.

Example 2.2.6 For $n=1$, any cycle $\mathrm{CH}^r(X,1)$ can be written as a finite sum $\sum_i (f_i, D_i)$ where $\mathrm{codim}_X(D_i) = r-1$, $f_i \in \mathbb{C}(D_i)^*$, and $\sum \mathrm{div}(f_i) = 0$.

Remark 2.2.7 The original construction by Bloch uses simplicial chains instead of cubical ones, in this case the role of the algebraic n-cube $\square^n$ is played by the algebraic simplex $\Delta^n = \mathrm{Spec}\ k[t_0,\ldots,t_n]/(\sum t_i - 1)$. The corresponding cycle complexes are quasi-isomorphic [Lev94]. In addition to Bloch's higher Chow groups, recently other cycle cohomology groups have been introduced for schemes of finite type over a field k. These include the *motivic cohomology* groups $H^q_{\mathcal{M}}(X,\mathbb{Z}(r))$ in the Voevodsky's sense of mixed motives (geometric motives $\mathrm{DM}(k)$, see [Voe00]), and the *bivariant cycle cohomology* groups $A_{rn}(X) = A_{rn}(k,X)$ of Friedlander-Voevodsky [FV00].

2.2.8 Motivic cohomology. Let $\mathrm{DM}_{\mathrm{gm}}(k)$ be the triangulated category of geometrical mixed motives over k. Denote by $M(X)$ the motive associated to X, and $\mathbb{Z}(r)$ the invertible Tate object. For $X \in \mathbf{Sm}(k)$, define the *Voevodsky's motivic cohomology* of X in degree q with twist p to be:

$$H^q_{\mathcal{M}}(X,\mathbb{Z}(p)) := \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(k)}(M(X),\mathbb{Z}(p)[q]).$$

The relation between higher Chow groups and motivic cohomology, in the smooth case, is given by the following comparison theorem:

Theorem 2.2.9 ([MVW06, Theorem 19.1]) *If X is smooth, then motivic cohomology agrees with Bloch's higher Chow groups*

$$H^q_{\mathcal{M}}(X, \mathbb{Z}(p)) \cong \mathrm{CH}^p(X, 2p-q).$$

In the special case when $q = 2p$, this is in particular a result on classical Chow groups (cycles modulo rational equivalence). Tensoring with $\mathbb{Q}$, we have the **Bloch-Grothendieck-Riemann-Roch theorem** [Blo86a]:

$$H^q_{\mathcal{M}}(X, \mathbb{Q}(p)) \cong K_{2p-q}(X)^{(p)}_{\mathbb{Q}}$$

with Adams eigenspaces of algebraic K-theory.

Remark 2.2.10 In general, when X is singular these isomorphisms fail, in this case higher Chow groups play the role of **motivic Borel-Moore homology** [FV00, Def. 9.1, Def. 4.3]:

$$H^{BM}_p(X, \mathbb{Z}(q)) = \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(k)}(\mathbb{Z}(q)[p], M_c(X)),$$

where $M_c(X)$ is the motive with compact support associated to X. By [MVW06, Proposition 19.18], we have

$$\mathrm{CH}^p(X, q) \cong H^{BM}_{2p+q}(X, \mathbb{Z}(p)).$$

But this situation can be remedied using *Hanamura's higher Chow cohomology groups* [Chap. 3, Sect. 3.2]. In [Han00], M. Hanamura replaces the singular variety by a diagram of smoth quasi-projective varieties called the cubical hyperresolution [GNPP88], together with Bloch's cycle complex to define its motivic cohomology, see also [Lev98].

2.2.11 Moving lemmas. For the construction of regulator and Abel-Jacobi morphism, in [KLM06] the authors use certain subcomplexes $\mathcal{Z}^p_{\mathbb{R}}(X, \bullet)$ of the cubical cycle complexes $\mathcal{Z}^p(X, \bullet)$ and a technique of moving lemma to show that this is quasi-isomorphic to Bloch's complex [KL07]. Here we give the definition:

Definition 2.2.12 [KLM06, 5.4] Let $C^p_{\mathbb{R}}(X, n)$ be the cycles $Z \in C^p(X, n)$ whose components intersect $X \times (T_{z_1} \cap \cdots \cap T_{z_i})$ and $X \times (T_{z_1} \cap \cdots \cap T_{z_i} \cap \partial^k \square^n)$ properly for all $1 \leq i \leq n$ and $1 \leq k \leq n$, and $D^p_{\mathbb{R}}(X, n) := C^p_{\mathbb{R}}(X, n) \cap D^p(X, n)$. We define the complex $\mathcal{Z}^p_{\mathbb{R}}(X, n) = C^p_{\mathbb{R}}(X, n)/D^p_{\mathbb{R}}(X, n)$.

The original moving lemma is due to Bloch and Levine, without subscript $\mathbb{R}$. This new version has been proved by Kerr-Lewis [KL07, Section 8.2]:

Theorem 2.2.13 (Moving lemmas)

- *The inclusion* $\mathcal{Z}^p_{\mathbb{R}}(X,\bullet)_{\mathbb{Q}} \hookrightarrow \mathcal{Z}^p(X,\bullet)_{\mathbb{Q}}$ *is a quasi-isomorphism in the derived category.*
- *For* $D \subset X$ *a closed subvariety of pure codimension* q*, the restriction*

$$\frac{\mathcal{Z}^p_{\mathbb{R}}(X,\bullet)_{\mathbb{Q}}}{\mathcal{Z}^{p-q}_{\mathbb{R}}(D,\bullet)_{\mathbb{Q}}} \to \mathcal{Z}^p_{\mathbb{R}}(X\setminus D,\bullet)_{\mathbb{Q}}$$

  gives a quasi-isomorphism.

Proof. See [KL07, 8.14 - 8.16]. The first statement is also valid for quasi-projective varieties.

2.2.14 The weight filtered spectral sequence [Lew16, Ex. 4.5] [KL07, Ex. 3.1]. Let X be a smooth, projective variety of dimension d over $\mathbb{C}$. Let $Y = Y_1 \cup \cdots \cup Y_N$ be a NCD with smooth components, and consider the smooth quasi-projective variety of the form $U = X \setminus Y$. For each integer $\ell \geq 0$, let $Y_{[\ell]}$ denote the disjoint union of t-fold intersections of the various components of Y, with corresponding semi-simplicial scheme $Y_{[\bullet]}$ with an augmentation to X, $Y_{[\bullet]} \to Y_{[0]} := X$ (see Example 3.1.12). Then, we have a third quadrant double complex $\mathcal{Z}^{i,j}_0(r) := \mathcal{Z}^{r+i}_{\mathbb{R}}(Y_{[-i]}, -j)$ for $i,j \leq 0$. The upper-right-hand can be seen as:

$$\begin{array}{ccccccc}
\longrightarrow & \mathcal{Z}^{r-2}(Y_{[2]}) & \xrightarrow{\mathrm{Gy}} & \mathcal{Z}^{r-1}(Y_{[1]}) & \xrightarrow{\mathrm{Gy}} & \mathcal{Z}^{r}(X) \\
& \partial\uparrow & & \partial\uparrow & & \partial\uparrow \\
\longrightarrow & \mathcal{Z}^{r-2}(Y_{[2]},1) & \xrightarrow[\mathrm{Gy}]{} & \mathcal{Z}^{r-1}(Y_{[1]},1) & \xrightarrow[\mathrm{Gy}]{} & \mathcal{Z}^{r}(X,1) \\
& \partial\uparrow & & \partial\uparrow & & \partial\uparrow \\
\longrightarrow & \mathcal{Z}^{r-2}(Y_{[2]},2) & \xrightarrow[\mathrm{Gy}]{} & \mathcal{Z}^{r-1}(Y_{[1]},2) & \xrightarrow[\mathrm{Gy}]{} & \mathcal{Z}^{r}(X,2) \\
& \uparrow & & \uparrow & & \uparrow
\end{array}$$

whose differentials are d vertically (∂ as coming from the definition of Bloch's cycle complex) and Gy (=Gysin) horizontally. The corresponding total complex $\mathbf{s}^\bullet\mathcal{Z}(r)$ with $D = \partial \pm \mathrm{Gy}$, comes associated with two spectral sequences:

$$E_2^{p,q} := H^p_{\mathrm{Gy}}\big(H^q_{\partial}(\mathcal{Z}^{\bullet,\bullet}_0(r))\big)$$

$${}'E_2^{p,q} := H^p_{\partial}\big(H^q_{\mathrm{Gy}}(\mathcal{Z}^{\bullet,\bullet}_0(r))\big).$$

According to [KL07, Section 3.1], the second spectral sequence together with Bloch's moving lemma $\mathcal{Z}^*(X,\bullet)/\mathcal{Z}^*(Y,\bullet) \xrightarrow{\text{qis}} \mathcal{Z}^*(U,\bullet)$, shows that:

$$Hm(\mathbf{s}^\bullet\mathcal{Z}(r)) = {}'E_2^{0,-m} = \mathrm{CH}(U,m).$$

The first spectral sequence gives $E_1^{i,j} = \mathrm{CH}^{r+i}(Y_{[-i]},-j)$ and

$$E_2^{i,j} = \frac{\ker\bigl(\mathrm{Gy}\colon \mathrm{CH}^{r+i}(Y_{[-i]},-j) \to \mathrm{CH}^{r+i+1}(Y_{[-i-1]},-j)\bigr)}{\mathrm{Gy}\bigl(\mathrm{CH}^{r+i-1}(Y_{[-i+1]},-j)\bigr)}.$$

The filtration on $\mathbf{s}^\bullet\mathcal{Z}(r)$ induces a "weight" filtration, which under change of indices becomes

$$\mathrm{Im}\bigl(\mathrm{CH}^r(X,m) \to \mathrm{CH}^r(U,m)\bigr) =: W_{-m}\mathrm{CH}^r(U,m) \subseteq$$

$$\cdots \subseteq W_0\mathrm{CH}^r(U,m) = \mathrm{CH}^r(U,m).$$

Proposition 2.2.15 *There is a third quadrant spectral sequence that converges to $\mathrm{CH}^r(U,m)$ with $E_1^{p,q} = \mathrm{CH}^{p+r}(Y^{[-p]},-q)$ (where $p+q=-m$), and $E_\infty^{0,-m} = W_{-n}\mathrm{CH}^r(U,m)$. The graded pieces are characterized by the injection (residue map)*

$$E_\infty^{-\ell-m,\ell} = \mathrm{Gr}_\ell^W\mathrm{CH}^r(U,m) \hookrightarrow \left\{ \begin{array}{c} \text{A subquotient of} \\ \mathrm{CH}^{r-\ell-m}(Y^{[\ell+m]},-\ell) \end{array} \right\}$$

for $-m \leq \ell \leq 0$.

2.3. Absolute Hodge cohomology

The construction of absolute Hodge cohomology is due to A. Beilinson [Bei86]. This theory generalizes Deligne-Beilinson cohomology in the sense that it includes the weight filtration. To explain this notion, we start with the definition of $\mathbb{A}$-mixed Hodge complex [HodgeIII], in particular a (polarizable) variant introduced by Beilinson in [Bei86].

Definition 2.3.1 Let $\mathbb{A}$ be a subring of $\mathbb{R}$. An **$\mathbb{A}$-mixed Hodge complex** consists of the following data:

– A bounded below complex $K_{\mathbb{A}}^\bullet$ of $\mathbb{A}$-modules, such that $H^p(K_{\mathbb{A}}^\bullet)$ is an $\mathbb{A}$-module of finite type for all p.

- A bounded below filtered complex $(K^\bullet_{\mathbb{A}\otimes\mathbb{Q}}, W)$ of $\mathbb{A}\otimes\mathbb{Q}$-vector spaces with an increasing filtration W, and an isomorphism $K^\bullet_{\mathbb{A}\otimes\mathbb{Q}} \xrightarrow{\sim} K^\bullet_{\mathbb{A}}\otimes\mathbb{Q}$ in $\mathbf{D}^+(\mathbb{A}\otimes\mathbb{Q})$.
- A bifiltered complex $(K^\bullet_{\mathbb{C}}, W, F)$ of $\mathbb{C}$-vector spaces with an increasing (respectively decreasing) filtration W (respectively F), and a filtered isomorphism $\alpha\colon (K^\bullet_{\mathbb{C}}, W) \to (K^\bullet_{\mathbb{A}\otimes\mathbb{Q}}, W)\otimes\mathbb{C}$ in $\mathbf{D}^+F(\mathbb{C})$.
- For every integer m, $\mathrm{Gr}^W_m K^\bullet_{\mathbb{A}\otimes\mathbb{Q}} \to (\mathrm{Gr}^W_m K^\bullet_{\mathbb{C}}, F)$ is a (polarizable) $\mathbb{A}\otimes\mathbb{Q}$-complex of weight m, i.e. the differentials $\mathrm{Gr}^W_m K^\bullet_{\mathbb{C}}$ are strictly compatible with the induced filtration F, and F induces a pure (polarizable) $A\otimes\mathbb{Q}$-Hodge structure of weight $m+r$ on $H^r(\mathrm{Gr}^W_m K^\bullet_{\mathbb{A}\otimes\mathbb{Q}})$ for $r\in\mathbb{Z}$.

A mixed $\mathbb{A}$-Hodge complex $K^\bullet_{\mathcal{H}}$ is given by a diagram (in the derived category) $K^\bullet_{\mathbb{A}} \xrightarrow{\alpha} (K^\bullet_{\mathbb{A}\otimes\mathbb{Q}}, W) \xrightarrow{\beta} (K^\bullet_{\mathbb{C}}, W, F)$. In this sense, by the definition of morphisms in derived categories, $K^\bullet$ gives rise to a diagram

$$\begin{tikzcd}[column sep=small] & {'K^\bullet_{\mathbb{A}\otimes\mathbb{Q}}} & & ({'K^\bullet_{\mathbb{C}}}, W) & \\ K^\bullet_{\mathbb{A}} \arrow[ur, "\alpha_1"] & & (K^\bullet_{\mathbb{A}\otimes\mathbb{Q}}, W) \arrow[ul, "\alpha_2"'] \arrow[ur, "\beta_1"] & & (K^\bullet_{\mathbb{C}}, W, F) \arrow[ul, "\beta_2"'] \end{tikzcd}$$

where α_j, β_j ($j=1,2$) are morphisms of complexes, α_2 is a quasi-isomorphism, and β_2 a filtered quasi-isomorphism. According to [Jan88, Theorem 2.2], by the work of Deligne and Beilinson, the construction of mixed $\mathbb{A}$-Hodge complexes is equivalent to the construction of mixed $\mathbb{A}$-Hodge structures. Moreover, by [Bei86, 3.11] the functor:

$$\mathbf{D}^b(\mathbf{MHS}^p) \to \mathbf{D}^b_{\mathbf{MHC}^p}$$

is an equivalence of triangulated categories, where $\mathbf{MHC}^p$ is the category of mixed Hodge complex with polarizable cohomology.

Definition 2.3.2 Let $K^\bullet_{\mathcal{H}}$ be an $\mathbb{A}$-mixed Hodge complex. Consider the following morphism

$$\begin{array}{ccc} K^\bullet_{\mathbb{A}} \oplus \widehat{W}_0 K^\bullet_{\mathbb{A}\otimes\mathbb{Q}} \oplus (\widehat{W}_0\cap F^0)K^\bullet_{\mathbb{C}} & \xrightarrow{(\alpha,\beta)} & {'K^\bullet_{\mathbb{A}\otimes\mathbb{Q}}} \oplus \widehat{W}_0\, {'K^\bullet_{\mathbb{C}}} \\ (\eta_{\mathbb{A}}, \eta_{\mathbb{Q}}, \eta_{\mathbb{C}}) & \mapsto & (\alpha_1\eta_{\mathbb{A}} - \alpha_2\eta_{\mathbb{Q}}, \beta_1\eta_{\mathbb{Q}} - \beta_2\eta_{\mathbb{C}}) \end{array}$$

where $\widehat{W}_\bullet := (\mathrm{Dec}\, W)_\bullet$ is Deligne's decalage filtration (1.2.7). The **absolute Hodge cohomology** is given by

$$H^\ell_{\mathcal{H}}(K^\bullet) := H^\ell\big(\mathrm{Cone}^\bullet(\alpha,\beta)[-1]\big) = H^\ell\big(R\Gamma_{\mathcal{H}}(K^\bullet)\big).$$

We henceforth assume $K^\bullet_{\mathbb{A}} = {}'K^\bullet_{\mathbb{A}}$, $K^\bullet_{\mathbb{C}} = {}'K^\bullet_{\mathbb{C}}$ with α_2, β_2 identity maps, and $K^\bullet_{\mathbb{A}\otimes\mathbb{Q}} = K^\bullet_{\mathbb{A}} \otimes \mathbb{Q}$ with α_1 the inclusion.

Remark 2.3.3 If $\mathbb{A} = \mathbb{Q}$, then α_1 is the identity. In this case there exists a diagram:

$$\begin{array}{c}
\mathrm{Cone}\big(K^\bullet_{\mathbb{Q}} \oplus \widehat{W}_0 K^\bullet_{\mathbb{Q}} \oplus F^0\widehat{W}_0 K^\bullet_{\mathbb{C}} \xrightarrow{(\alpha,\beta)} K^\bullet_{\mathbb{Q}} \oplus \widehat{W}_0 K^\bullet_{\mathbb{C}}\big)[-1] \\
\mathrm{qis}\big\uparrow \\
C^\bullet_{\mathcal{H}} := \mathrm{Cone}\big(\widehat{W}_0 K^\bullet_{\mathbb{Q}} \oplus F^0\widehat{W}_0 K^\bullet_{\mathbb{C}} \xrightarrow{\beta_1-\beta_2} \widehat{W}_0 K^\bullet_{\mathbb{C}}\big)[-1] \\
\big\downarrow D \\
C^\bullet_{\mathcal{D}} := \mathrm{Cone}\big(K^\bullet_{\mathbb{Q}} \oplus F^0 K^\bullet_{\mathbb{C}} \xrightarrow{\beta_1-\beta_2} K^\bullet_{\mathbb{C}}\big)[-1]
\end{array}$$

given by $(\eta_1, \eta_2, \eta_3) \mapsto (\eta_1, \eta_1, \eta_2, 0, \eta_3)$. The differential morphism D between absolute Hodge and Deligne complex is given by $D(\eta_1, \eta_2, \eta_3) = (-d\eta_1, -d\eta_2, d\eta_3 + \beta_1\eta_1 - \beta_2\eta_2)$, which simply forgets the weights.

2.3.4 Hodge realizations. According to M. Levine [Lev98] and A. Huber [Hub00], there is a realization functor of triangulated categories

$$R_{\mathcal{H}} \colon \mathrm{DM}_{\mathrm{gm}}(k) \to \mathbf{D}^b(\mathbf{MHS})$$

that by functoriality, induces a regulator morphism:

$$\mathrm{cl}_{\mathcal{H}} \colon H^r_{\mathcal{M}}(X, \mathbb{Q}(m)) \to H^r_{\mathcal{H}}(X, \mathbb{Q}(m))$$

which is compatible with the localization sequence, and weight filtrations. The explicit constructions, using currents, were made in [KLM06] and [KL07], and we will present them in the following sections.

2.3.5 The smooth projective case. Let X be a smooth and projective variety over $\mathbb{C}$. To produce an explicit mixed Hodge complex that computes its absolute Hodge cohomology, consider the following system [Jan88, 2.7]:

$$\begin{aligned}
({}'K^\bullet_{\mathbb{Z}} =) K^\bullet_{\mathbb{Z}} &:= C(X, \mathbb{Z}(p))[2p]^\bullet = C^{2p+\bullet}(X, \mathbb{Z}(p)), \\
K^\bullet_{\mathbb{Z}\otimes\mathbb{Q}} &:= C^{2p+\bullet}(X, \mathbb{Z}(p)) \otimes \mathbb{Q}, \\
({}'K^\bullet_{\mathbb{C}} =) K^\bullet_{\mathbb{C}} &:= \mathcal{D}(X)(p)[2p]^\bullet,
\end{aligned}$$

where $K^\bullet_{\mathbb{Z}\otimes\mathbb{Q}} \xrightarrow{\beta_1} K^\bullet_{\mathbb{C}}$ is given by integration: $(2\pi i)^p\gamma \mapsto (2\pi i)^p \int_\gamma$.

– $F^\bullet$ is the Hodge filtration on $\mathcal{D}_X^\bullet$ twisted by p.

– $W_\bullet$ is the stupid weight filtration

$$W_r K_-^\bullet := \begin{cases} K_-^\bullet & r \geq 0 \\ 0 & r < 0. \end{cases}$$

This filtration induces a filtered morphism $(K_{\mathbb{Q}}^\bullet, W_\bullet) \otimes \mathbb{C} \xrightarrow{\text{qis}} (K_{\mathbb{C}}^\bullet, W_\bullet)$ given by the Poincaré lemma. Then, the décalage filtration is

$$\widehat{W}_0 K^m_{A\otimes\mathbb{Q}} = \ker\Big\{W_{-m} K^m_{A\otimes\mathbb{Q}} \to \frac{K^{m+1}_{A\otimes\mathbb{Q}}}{W_{-m-1}K^{m+1}_{A\otimes\mathbb{Q}}}\Big\}$$

$$= \begin{cases} K^m_{A\otimes\mathbb{Q}}, & m < 0 \\ \ker(\partial) \subset K^0_{A\otimes\mathbb{Q}}, & m = 0 \\ 0, & m > 0. \end{cases}$$

Explicitly, the complex $\mathrm{Cone}\big(K_{\mathbb{Q}}^\bullet \oplus \widehat{W}_0 K_{\mathbb{Q}}^\bullet \oplus F^0 \widehat{W}_0 K_{\mathbb{C}}^\bullet \to K_{\mathbb{Q}}^\bullet \oplus \widehat{W}_0 K_{\mathbb{C}}^\bullet\big)[-1]$ becomes the following complexes in each case:

$$\begin{cases} C^{2p+\bullet}(X, \mathbb{A}(p)) \oplus C^{2p+\bullet-1}(X, \mathbb{A}(p)) \otimes_{\mathbb{Z}} \mathbb{Q}, & \bullet > 1 \\[2mm] C^{2p+1}(X, \mathbb{A}(p)) \oplus C^{2p}(X, \mathbb{A}(p)) \otimes_{\mathbb{Z}} \mathbb{Q} \oplus \{\ker(d) \subset \mathcal{D}^{2p}(X)\}, & \bullet = 1 \\[2mm] C^{2p}(X, \mathbb{A}(p)) \oplus \{\ker(\partial) \subset C^{2p}(X, \mathbb{A}(p)) \otimes_{\mathbb{Z}} \mathbb{Q}\} & \\ \oplus \{\ker(d) \subset F^p\mathcal{D}^{2p}(X)\} \oplus C^{2p-1}(X, \mathbb{A}(p)) \otimes_{\mathbb{Z}} \mathbb{Q} \oplus \mathcal{D}^{2p-1}(X), & \bullet = 0 \\[2mm] C^{2p+\bullet}(X, \mathbb{A}(p)) \oplus C^{2p+\bullet}(X, \mathbb{A}(p)) \otimes_{\mathbb{Z}} \mathbb{Q} \oplus F^p\mathcal{D}^{2p+\bullet}(X) & \\ \oplus C^{2p+\bullet-1}(X, \mathbb{A}(p)) \otimes_{\mathbb{Z}} \mathbb{Q} \oplus \mathcal{D}^{2p+\bullet-1}(X), & \bullet < 0. \end{cases}$$

For $q \in \mathbb{Z}$, its q^{th}-cohomology is $H^{2p+q}_{\mathcal{H}}(X, \mathbb{A}(p))$. By forgetting all $\widehat{W}_0$'s in the definition of the Cone complex, we have a morphism

$$H^{2p+q}_{\mathcal{H}}(X, \mathbb{A}(p)) \to H^{2p+q}_{\mathcal{D}}(X, \mathbb{A}(p)).$$

This is an isomorphism in degrees $q \leq 0$.

2.4. The regulator morphism: KLM-formula

The KLM-formula is a morphism of complexes inducing the Bloch-Beilinson regulator morphism with rational coefficients, where the Deligne cohomology is computed by a complex of 3-terms [KLM06]. The Abel-Jacobi morphism for higher Chow groups $\mathrm{CH}^q(X,m)$ generalizes the classical Abel-Jacobi morphism on the Chow groups $\mathrm{CH}^q(X)$, this is a morphism from higher Chow cycles homologous to zero to the Griffiths intermediate Jacobian.

2.4.1 The currents. Let $(z_1,\dots,z_m)\in\square^m$ be the affine coordinates. On $\square^m$ we have the following currents [KLM06, page 383]:

$$\Omega_\square = \int_{\square^m} \bigwedge_{j=1}^{m} d\log z_j$$

$$\mathbf{T}_\square = (2\pi i)^m \int_{[-\infty,0]^m} (-)$$

$$R_\square = \Bigg[\int_{\square^m} \log z_1 \bigwedge_{j=1}^{r} d\log z_j - (2\pi i)\int_{[-\infty,0]\times\square^{m-1}} \log z_2 \bigwedge_{j=3}^{r} d\log z_j + \cdots \\ + (-2\pi i)^m \int_{[-\infty,0]^{m-1}\times\square^1} d\log z_m \Bigg].$$

Consider a cycle $\alpha\in\mathcal{Z}^r(X\times\square^m)$ in general position. Let $\pi_1\colon|\alpha|\to X$, $\pi_2\colon|\alpha|\to\square^m$ be the projections. Then, we have the corresponding currents[1]:

$$\begin{aligned} R_\alpha &= (\pi_{1*}\circ\pi_2^*)\ R_\square, \\ \Omega_\alpha &= (\pi_{1*}\circ\pi_2^*)\ \Omega_\square, \\ T_\alpha &= (\pi_{1*}\circ\pi_2^*)\ T_\square. \end{aligned}$$

Recall that in the Deligne cohomology complex

$$C^\bullet_{\mathcal{D}} = \mathrm{Cone}\{C^{2p+\bullet}(X,\mathbb{Z}(p))\oplus F^p\mathcal{D}_X^{2p+\bullet}(X)\to\mathcal{D}_X^{2p+\bullet-1}(X)\}[-1]$$

the differential D is given by

$$\begin{aligned} D\big((2\pi i)^{p-r}(\mathbf{T}_\eta,\Omega_\eta,R_\eta)\big) &= (2\pi i)^{p-r}(dT_\eta, d\Omega_\eta, \mathbf{T}_\eta-\Omega_\eta-dR_\eta) \\ &= (2\pi i)^{p-r+1}(\mathbf{T}_{\partial\eta},\Omega_{\partial\eta},R_{\partial\eta}); \end{aligned}$$

[1] In the clasicall case when $m=0$, we have that $R_\alpha=0$.

the resulting cohomology at $\bullet = -r$ is $H_{\mathcal{D}}^{p-r+1}(X, \mathbb{Z}(p))$. To guarantee that the currents are defined, we have to restrict to the subcomplex $\mathcal{Z}_{\mathbb{R}}^{p}(X, \bullet)$ of real cycles. Then, we have the main result of [KLM06]:

Theorem 2.4.2 ([KLM06, 5.5]) *There is a morphism of complexes*

$$\mathrm{reg}_X \colon \mathcal{Z}_{\mathbb{R}}^{p}(X, \bullet) \to C_{\mathcal{D}}^{2p+\bullet}(X, \mathbb{Q}(p)),$$

induced by $\eta \mapsto (2\pi i)^{p-r}(\mathbf{T}_\eta, \Omega_\eta, R_\eta)$.

This morphism of complexes is called the **regulator morphism** in the smooth and projective case [KLM06]. Taking the corresponding cohomology groups, we have the Bloch's cycle class morphism [Blo86b]:

$$\mathrm{cl}_{r,m} \colon \mathrm{CH}^r(X, m) \to H_{2d-2r+m}^{\mathcal{D}}(X, \mathbb{Z}(r-d)) \xrightarrow{\cong} H_{\mathcal{D}}^{2r-m}(X, \mathbb{Z}(r)).$$

Recall that Deligne cohomology sits in an exact sequence

$$0 \to \frac{H^{2q-n-1}(X, \mathbb{C})}{F^q H^{2q-n-1}(X, \mathbb{C}) + H^{2q-n-1}(X, \mathbb{Z}(q))} \to H_{\mathcal{D}}^{2q-n}(X, \mathbb{Z}(q))$$

$$\to F^q H^{2q-n}(X, \mathbb{C}) \bigcap H^{2q-n}(X, \mathbb{Z}(q)) \longrightarrow 0.$$

Then, we have the fundamental class morphism

$$\begin{aligned} \mathrm{cl}_{r,m} \colon \mathrm{CH}^r(X, m) \to \mathrm{Hdg}^{r,m}(X) \quad &:= \quad \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}(0), H^{2r-m}(X, \mathbb{Q}(r))) \\ &\cong \quad F^q H^{2q-n}(X, \mathbb{C}) \bigcap H^{2q-n}(X, \mathbb{Z}(q)). \end{aligned}$$

If we define the nullhomologous higher cycles by

$$\mathrm{CH}_{\mathrm{hom}}^q(X, n) = \ker\{\mathrm{CH}^q(X, n) \to H_{\mathcal{D}}^{2q-n}(X, \mathbb{Z}(q)) \to H^{2q-n}(X, \mathbb{Z}(q))\}.$$

Following [KLM06], the induced map

$$\mathrm{AJ}_{r,m} \colon \mathrm{CH}_{\mathrm{hom}}^r(X, m) \to \mathrm{Jac}^{r,m}(X) := \frac{F^{d-r+1} H^{2d-2r+m+1}(X, \mathbb{C})^\vee}{H_{2d-2r+m+1}(X, \mathbb{Z}(d-r))},$$

is the **Abel-Jacobi morphism**, and is given as follows (in terms of currents). Explicitly, if $\alpha \in \mathrm{CH}^r(X, m)$ is homologous to zero, such that each irreducible component intersects all real faces properly, the formula for the Abel-Jacobi map is given by the following current:

$$\begin{aligned}\alpha \quad \mapsto \frac{1}{(2\pi i)^{d-r+m}} & \Bigg[\int_{\alpha \setminus \pi_2^{-1}[-\infty,0]\times\square^{m-1}} \pi_2^*\big((\log z_1)\; d\log z_2 \wedge \cdots \wedge d\log z_m\big) \wedge \pi_1^*(\omega) \\ (-2\pi i) & \int_{\alpha\cap\pi_2^{-1}[-\infty,0]\times\square^{m-1}\setminus\pi_2^{-1}[-\infty,0]^2\times\square^{m-2}} \pi_2^*\big((\log z_2)\; d\log z_3 \wedge \cdots \wedge d\log z_m\big) \wedge \pi_1^*(\omega) \\ + \cdots + (-2\pi i)^{m-1} & \int_{\alpha\cap\pi_2^{-1}[-\infty,0]^{m-1}\times\square^1\setminus\alpha\cap\pi_2^{-1}[-\infty,0]^m} \pi_2^*(\log z_m) \wedge \pi_1^*(\omega) \\ (-2\pi i)^m & \int_\gamma \pi_1^*(\omega)\Bigg]\end{aligned}$$

Replacing the n-cube $\square^n$ by the n-simplex Δ^n yields similar operators T_α, Ω_α and R_α; the description of these operatos appears in [KLL18].

Remark 2.4.3 According to [KLM06], the projection of the $\mathrm{cl}_{r,m}$ morphism to the real Deligne cohomology, i.e. the composition

$$\mathrm{CH}^r(X,m) \xrightarrow{\mathrm{cl}_{r,m}} H^{2r-n}_{\mathcal{D}}\big(X,\mathbb{Z}(r)\big) \xrightarrow{\pi_{\mathbb{R}}} H^{2r-n}_{\mathcal{D}}\big(X,\mathbb{R}(r)\big)$$

agrees with the regulator defined by Goncharov [Gon95]; this is the **real regulator morphism**.

Example 2.4.4 Let X be an elliptic curve in $\mathbb{P}^2_{\mathbb{C}}$. This is smooth, projective curve of genus one. In this case, $\mathrm{CH}^1_{\mathrm{hom}}(X,0)$ are the 0-cycles of degree zero and

$$\mathrm{Jac}^{1,0}(X) = \frac{H^{1,0}(X)^\vee}{H_1(X,\mathbb{Z})} \cong \frac{\mathbb{C}}{\mathbb{Z}^2}.$$

For a cycle $\alpha \in \mathrm{CH}_0(X)_{\deg\,0}$ we can write it in the form $\alpha = \sum_j (p_j - q_j)$. Consider any real 1-chain γ on X such that $\partial\gamma = \alpha$. Note that $H^{1,0}(X) = \mathbb{C}\omega$, where $\omega = dx/y = dx/\sqrt{h(x)}$. Then

$$\mathrm{AJ}_{1,0}(\alpha)(\omega) = \int_\gamma \frac{dx}{\sqrt{h(x)}} = \sum_j \int_{q_j}^{p_j} \frac{dx}{\sqrt{h(x)}},$$

is the classical elliptic integral.

Example 2.4.5 Let X be a complex surface.Then, the classes in $\mathrm{CH}^2(X,1)$ are represented by $\big\{\alpha = \sum_i (f_i, C_i) \mid \sum_i \mathrm{div}(f_i) = 0\big\}$, where the C_i's are curves. The cycle class morphism $\mathrm{cl}_{2,1}\colon \mathrm{CH}^2(X,1) \to H^3_{\mathcal{D}}(X,\mathbb{Z}(2))$, induces an Abel-Jacobi morphism

$$\mathrm{AJ}_{2,1}\colon \mathrm{CH}^2_{\mathrm{hom}}(X,1) \to \frac{\{H^{2,0}(X) \oplus H^{1,1}(X)\}^\vee}{H_2(X,\mathbb{Z})},$$

given as follows. Let $\alpha \in \mathrm{CH}^2(X,1)$, with $\alpha = \sum_j (f_i, C_i)$. The morphisms can be seen as $f_i\colon C_i \to \mathbb{P}^1$, and consider $\gamma_i := f_i^{-1}([0,\infty])$. The condition $\sum_i \mathrm{div}(f_i) = 0$ implies that $\gamma := \sum_i \gamma_i$ defines a 1-cycle. Since $\alpha \in \mathrm{CH}^2_{\mathrm{hom}}(X,1)$, there is a real chain ζ in X, such that $\gamma = \partial(\zeta)$. For $\omega \in H^{2,0}(X) \oplus H^{1,1}(X)$, we have:

$$\mathrm{AJ}_{2,1}(\alpha) = \frac{1}{(2\pi i)}\Big(\sum_i \int_{C_i \setminus \gamma_i} \omega \, \log(f_i) + 2\pi i \int_\zeta \omega\Big).$$

2.5. The smooth and quasi-projective case

Let U be a smooth, quasi-projective complex variety. Consider a good compactification $U \hookrightarrow X$ of U, which means that X is a smooth projective variety and $Y := X \setminus U = \bigcup_{i=1}^N Y_i$ is a normal-crossing divisor (NCD) with smooth components (and smooth intersections of all orders). In [KL07], Kerr and Lewis describe the *Hodge cycle class map*:

$$\mathrm{cl}^{r,m}_{\mathcal{H}}\colon \mathrm{CH}^r(U,m) \to H^{2r-m}_{\mathcal{H}}(U,\mathbb{Q}(r))$$

in this case they reduce everything to the smooth projective case [KLM06] through weight filtered spectral sequence on both sides. The regulator morphism is given in terms of double complexes, evaluated on the graded pieces of Gysin espectral sequence.

2.5.1 The mixed Hodge complex. Let $Y_{[\ell]}$ denote the disjoint union of t-fold intersections of various components of Y, with corresponding hypercovering $Y_{[\bullet]} \to Y \hookrightarrow X$. Then, we have defined double complexes

$$\mathcal{D}(r)^{i,j} := \mathcal{D}^{2r+2i+j}(Y_{[-i]})(r+i),$$
$$C(r)^{i,j} := C(r)^{2r+2i+j}\big(Y_{[-i]}, \mathbb{A}(r+i)\big),$$

with differentials Gy (=Gysin, horizontal) in both cases, and d repectively ∂ (vertical). Consider the associated total complexes

$$\mathbf{s}^\bullet\mathcal{D}(r) := \bigoplus_{i(\leq 0)} \mathcal{D}(r)^{i,\bullet-i} = \bigoplus \mathcal{D}^{2r+i+\bullet}(Y_{[-i]}),$$

$$\mathbf{s}^\bullet C(r) := \bigoplus_{i(\leq 0)} C(r)^{i,\bullet-i} = \bigoplus C^{2r+i+\bullet}\big(Y_{[-i]}, \mathbb{A}(r+i)\big),$$

with differentials $D = d \pm \mathrm{Gy}$ and $D = \partial \pm \mathrm{Gy}$. The "weight" filtrations are

$$'W^\ell\big(\mathbf{s}^k \mathcal{D}(r)\big) := \bigoplus_{\ell \leq i(\leq 0)} \mathcal{D}(r)^{i,k-i}$$

$$'W^\ell\big(\mathbf{s}^k C(r)\big) := \bigoplus_{\ell \leq i(\leq 0)} C(r)^{i,k-i}$$

and Hodge filtration is given by $F^q \mathbf{s}^k \mathcal{D}(r) := \bigoplus_i F^{q+r+i} \mathcal{D}^{2r+2i+j}(Y_{[-i]})$. This is the basic principle to endow the cohomology of a variety with a mixed Hodge structure, the necessary filtrations must be defined on the level of complexes to give rise to filtrations on cohomology. Then, we have

$$H^*(F^q \mathbf{s}^\bullet \mathcal{D}(r)) \cong F^{q+r} H^{*+2r}_{DR}(U, \mathbb{C}), \quad H^*(\mathbf{s}^\bullet C(r)) \cong H^{*+2r}(U, \mathbb{Q}(r)).$$

Recall that for the filtered complex $(\mathbf{s}^\bullet \mathcal{D}(r), {}'W)$, the décalage filtration $\mathrm{Dec}({}'W)$ of ${}'W$ is given by:

$$(\mathrm{Dec}\ {}'W)^\ell \mathbf{s}^k \mathcal{D}(r) = \ker\Big\{{}'W^{\ell+k} \mathbf{s}^k \mathcal{D}(r) \xrightarrow{D} \frac{\mathbf{s}^{k+1}\mathcal{D}(r)}{{}'W^{\ell+k+1}\mathbf{s}^{k+1}\mathcal{D}(r)}\Big\}.$$

This is other filtration on the complex $\mathbf{s}^\bullet \mathcal{D}(r)$, and we have:

Proposition 2.5.2 *The décalage filtration of ${}'W$ on $\mathbf{s}^k \mathcal{D}(r)$ is the canonical filtration.*

Proof. By definition of décalage filtration

$$\begin{aligned}
(\mathrm{Dec}\ {}'W)^\ell \mathbf{s}^k \mathcal{D}(r) &= \ker\Big\{ \bigoplus_{\ell+k \leq i(\leq 0)} \mathcal{D}(r)^{i,k-i} \xrightarrow{D} \bigoplus_{(0 \geq)\ell+k+1 > i} \mathcal{D}(r)^{i,k+1-i} \Big\} \\
&= \ker\Big\{ \mathcal{D}(r)^{\ell+k,-\ell} \xrightarrow{d} \mathcal{D}(r)^{\ell+k,-\ell+1} \Big\} \\
&\quad \oplus \bigoplus_{\ell+k < i(\leq 0)} \mathcal{D}^{2r+i+k}(Y_{[-i]})(r+i) \\
&= \mathbf{s}^k\{\tau_{\leq -\ell} \mathcal{D}(r)^{i,\bullet}\}.
\end{aligned}$$

□

The usual weights on cohomology are given by the "weight" (decreasing) filtration ${}'W^\bullet$, under the change of indices $W_\bullet (:= {}'W^{-\bullet})$, and we have

$$\begin{aligned}
{}'W^{-\ell} H^k(U, \mathbb{Q}(r)) &:= \mathrm{Im}\big\{H^k({}'W^{-\ell}\mathbf{s}^\bullet C(r)) \to H^k(\mathbf{s}^\bullet C(r))\big\} \\
&=: W_{2r+k+\ell} H^{2r+k}(U, \mathbb{Q}).
\end{aligned}$$

The action of décalage filtration is the degeneration of the spectral sequence in the first page [HodgeII]. The Gysin spectral sequence with 0^{th}-page

$$E_0^{i,j} := \mathrm{Gr}^i_{'W}\mathbf{s}^{i+j}\mathcal{D}(r) \cong \mathcal{D}(r)^{i,j},$$

by [HodgeIII] degenerates at the stage E_2, hence the following spectral sequence

$$\widehat{E}_0^{i',j'} := \mathrm{Gr}^{i'}_{\mathrm{Dec}('W)}\mathbf{s}^{i'+j'}\mathcal{D}(r)$$

degenerates at $\widehat{E}_1$ by the isomorphism $\widehat{E}_r^{-j,i+2j} \xrightarrow{\cong} E_{r+1}^{i,j}$. Both converge to $H^*\mathbf{s}^\bullet\mathcal{D}(r)$. The compatibility of the total differential $\mathbb{D}$ with the filtration $\mathrm{Dec}('W)^\bullet$, gives us the relation between $'W$ and $\mathrm{Dec}('W)$ is

$$H^k\big(\mathrm{Dec}('W)^{-\ell-k}\mathbf{s}^\bullet\mathcal{D}(r)\big) \cong \mathrm{Im}\big\{H^k\big(\mathrm{Dec}('W)^{-\ell-k}\mathbf{s}^\bullet\mathcal{D}(r)\big) \to H^k\big(\mathbf{s}^\bullet\mathcal{D}(r)\big)\big\}.$$

Writing $\widehat{W}_0 = \mathrm{Dec}('W)^0$, the compatibility of spectral sequences implies, when $k = -\ell$, that

$$H^k(\widehat{W}_0\mathbf{s}^\bullet\mathcal{D}(r)) \cong W_{2r}H^{2r+k}(U, \mathbb{C}).$$

Corollary 2.5.3 *The following system $K^\bullet$ given by:*

- $(K^\bullet_{\mathbb{Q}}, W) = (\mathbf{s}^\bullet C(r), W_\bullet)$,
- $(K^\bullet_{\mathbb{C}}, W, F) = (\mathbf{s}^\bullet\mathcal{D}(r), W_\bullet, F^\bullet)$

defines a mixed Hodge complex associated to X, where $W_\bullet = {}'W^{-\bullet}$ and β_1 is given by integration on each $Y_{[-i]}$.

Definition 2.5.4 The absolute Hodge cohomology of U is given by the mixed Hodge complex $K^\bullet$, i.e. $H^{2p+q}_{\mathcal{H}}(U, \mathbb{Q}(p)) := H^q_{\mathcal{H}}(K^\bullet)$.

The cone complex associated to $K^\bullet$, by definition is:

$$\mathbb{Q}^\bullet_{\mathcal{H}}(r) := \mathrm{Cone}\big\{\mathbf{s}^\bullet C(r) \bigoplus \widehat{W}_0\mathbf{s}^\bullet C(r) \otimes \mathbb{Q} \bigoplus (\widehat{W}_0 \cap F^0)\mathbf{s}^\bullet\mathcal{D}(r)$$

$$\to \mathbf{s}^\bullet C(r) \otimes \mathbb{Q} \bigoplus (\widehat{W}_0 \cap F^0)\mathbf{s}^\bullet\mathcal{D}(r)\big\}[-1]$$

where $\widehat{W}_0\mathbf{s}^\bullet(r) = \ker\Big\{'W^k\mathbf{s}^k(r) \to \frac{\mathbf{s}^{k+1}(r)}{'W^{k+1}\mathbf{s}^k(r)}\Big\}$. By remark (2.3.3), the absolute Hodge cohomology is computed by the k^{th}-cohomology of the complex

$$\begin{aligned}
\mathbb{Q}^\bullet_{\mathcal{H}}(r) &:= \mathrm{Cone}\big(\widehat{W}_0\mathbf{s}^\bullet C(r) \oplus F^0\widehat{W}_0\mathbf{s}^\bullet\mathcal{D}(r) \longrightarrow \widehat{W}_0\mathbf{s}^\bullet\mathcal{D}(r)\big)[-1] \\
&= \mathrm{Cone}\left\{\begin{array}{c} \mathbf{s}^\bullet(\tau_{j\le 0}C(r)) \\ \oplus[(F^0\mathbf{s}^\bullet\mathcal{D}(r)) \cap \mathbf{s}^\bullet(\tau_{j\le 0}\mathcal{D}(r))] \end{array} \xrightarrow{\beta_1-\beta_2} \mathbf{s}^\bullet(\tau_{j\le 0}\mathcal{D}(r))\right\} \\
&= \mathbf{s}^\bullet\mathcal{H}(r)^{\bullet,\bullet}
\end{aligned}$$

where $\mathbf{s}^\bullet\mathcal{H}(r)^{\bullet,\bullet}$ is the total complex associated to the double complex

$$\mathcal{H}(r)_0^{i,j} := \begin{cases} 0, & j > 1 \\ \ker(d) \subset \mathcal{D}^{2(r+1)}(Y_{[-i]}), & j = 1 \\ \ker(\partial) \subset C^{2(r+i)}\big(Y_{[-i]}, \mathbb{Q}(r+i)\big) \bigoplus & \\ \ker(d) \subset F^0\mathcal{D}^{2(r+1)}(Y_{[-i]}) \bigoplus \mathcal{D}^{2r+2i-i}(Y_{[-i]}), & j = 0 \\ C_{\mathcal{D}}^{2r+2i+j}\big(Y_{[-i]}, \mathbb{Q}(r+i)\big), & j < 0. \end{cases}$$

This double complex is third quadrant, with differentials D (cone differential, vertical) and Gysin (horizontal), and is zero if $i > 0$. The elements of $\mathcal{H}(r)^{i,j}$ can be written as (a, b, c), with a, b both zero if $j = 1$.

2.5.5 The regulator morphism. As $Y_{[-i]}$ is smooth and projective, the KLM-formula gives a realization morphisms of double complexes:

$$\begin{array}{rcl} \mathcal{Z}_0^{i,j}(r) := \mathcal{Z}_{\mathbb{R}}^{r+i}(Y_{[-i]}, -j) & \to & \mathcal{H}(r)_0^{i,j} \\ \xi & \mapsto & (-2\pi i)^{r+j}\big((2\pi i)^{-j}T_\xi, \Omega_\xi, R_\xi\big). \end{array}$$

via the quasi-isomorphism $C^\bullet_{\mathcal{H}}(Y_{[-i]}, \mathbb{Q}(r)) \to C^\bullet_{\mathcal{D}}(Y_{[-i]}, \mathbb{Q}(r))$. By considering the total complex of both sides, the KLM- formula defines a regulator morphism on the level of complexes for U

$$\mathrm{reg}_U \colon \mathbf{s}^\bullet\big(\mathcal{Z}^{\bullet,\bullet}(r)\big) =: \mathcal{Z}_U^\bullet(r) \to \mathcal{H}_U^\bullet(r) := \mathbf{s}^\bullet\big(\mathcal{H}^{\bullet,\bullet}(r)\big).$$

The weight filtered cycle-class morphism is given by the following spectral sequence:

Proposition 2.5.6 *There is a (Gysin) spectral sequence that converges to $H^*(\mathbf{s}^\bullet\mathcal{H}(r)^{\bullet,\bullet}) \cong H_{\mathcal{H}}^{2r+*}(U, \mathbb{Q}(r))$, with $\mathcal{H}(r)_0^{i,j} := \mathcal{H}(r)^{i,j}$ and $d_0 := D$. This weight spectral sequence induces a weight filtration with graded pieces:*

$$\mathrm{Gr}_{'W}^{-i} H^{i+j}(\mathbf{s}^\bullet\mathcal{H}(r)) = \mathrm{Gr}_j^W H_{\mathcal{H}}^{2r+i+j}(U, \mathbb{Q}(r)) \cong \mathcal{H}(r)_\infty^{i,j}.$$

The total cohomologies give us the cycle-class morphisms

$$\mathrm{cl}_{\mathcal{H}}^{r,m} \colon \mathrm{CH}^r(U, m) \cong H_{\mathcal{M}}^{2r-m}(U, \mathbb{Q}(r)) \ \to \ H_{\mathcal{H}}^{2r-m}(U, \mathbb{Q}(r))$$

with graded pieces

$$\mathrm{Gr}_{-\ell}^W(\mathrm{cl}_{\mathcal{H}}^{r,m}) \colon \mathcal{Z}(r)_\infty^{-\ell-m,\ell} \ \to \ \mathcal{H}(r)_\infty^{-\ell-m,\ell}.$$

To give a precise description of the Abel-Jacobi morphism, Kerr and Lewis describe explicitly these graded pieces. According to [KL07], there is a weight spectral sequence on singular cohomology, that degenerates at the second page. Consider the following short exact sequence of MHS:

$$0 \to W_{\ell-1} \to W_0 \to \mathrm{Gr}_W^{0,\ell-1} \to 0$$

applied on the cohomology $H^{2r-m-1}(U, \mathbb{Q}(r))$, and define

$$\Xi_\ell := \mathrm{Image}\left(\begin{array}{c} \mathrm{Hom}_{\mathrm{MHS}}\big(\mathbb{Q}(0), \mathrm{Gr}^W_{0,\ell-1} H^{2r-m-1}(X \setminus Y, \mathbb{Q}(r))\big) \\ \downarrow \\ \mathrm{Ext}^1_{\mathrm{MHS}}\big(\mathbb{Q}(0), \mathrm{Gr}^W_{\ell-1} H^{2r-m-1}(X \setminus Y, \mathbb{Q}(r))\big) \end{array}\right)$$

Proposition 2.5.7 ([KL07, Prop. 2.7]) *The graded pieces of absolute Hodge cohomology are given by*

$$\mathcal{H}(r)_\infty^{-m-\ell,\ell} \cong \frac{\mathrm{Ext}^1_{\mathrm{MHS}}\big(\mathbb{Q}(0), \mathrm{Gr}^W_{j-1} H^{2r-m-1}(X \setminus Y, \mathbb{Q}(r))\big)}{\Xi_j}$$

for $-m \le \ell < 0$, and for $\ell = 0$ there is a short exact sequence:

$$0 \longrightarrow \frac{\mathrm{Ext}^1_{\mathrm{MHS}}\big(\mathbb{Q}(0), \mathrm{Gr}^W_{-1} H^{2r-m-1}(X \setminus Y, \mathbb{Q}(r))\big)}{\Xi_0}$$

$$\longrightarrow \mathcal{H}(r)_\infty^{-m,0} \longrightarrow \mathrm{Hom}_{\mathrm{MHS}}\big(\mathbb{Q}(0), H^{2r-m}(X \setminus Y, \mathbb{Q}(r)\big) \longrightarrow 0.$$

2.5.8 On the other hand, the Cone presentation of $\mathbf{s}^\bullet \mathcal{H}(r)^{\bullet,\bullet}$, it gives a short exact sequence:

$$0 \longrightarrow \mathrm{Ext}^1_{\mathrm{MHS}}\big(\mathbb{Q}(0), H^{2r-m-1}(U, \mathbb{Q}(r))\big) \longrightarrow H^{2r-m}_{\mathcal{H}}(U, \mathbb{Q}(r))$$
$$\longrightarrow \mathrm{Hom}_{\mathrm{MHS}}\big(\mathbb{Q}(0), H^{2r-m}(U, \mathbb{Q}(r))\big) \longrightarrow 0.$$

Define:

$$\mathrm{CH}^r_{\mathrm{hom}}(U, m) := \mathrm{Ker}\big\{\mathrm{CH}^r(U, m) \to \mathrm{Hom}_{\mathrm{MHS}}\big(\mathbb{Q}(0), H^{2r-m}(U, \mathbb{Q}(r))\big)\big\}.$$

The following result is the Abel-Jacobi in the quasi-projective case:

Theorem 2.5.9 *There is an Abel-Jacobi morphism*

$$\mathrm{AJ}_{r,m}\colon \mathrm{CH}^r_{\mathrm{hom}}(X\setminus Y,m)\to \mathrm{Ext}^1_{\mathrm{MHS}}\big(\mathbb{Q}(0),H^{2r-m-1}(X\setminus Y,\mathbb{Q}(r))\big)$$

such that in the graded pieces $\mathrm{Gr}^W_j(\mathrm{AJ}_{r,m})$ *look like:*

$$\begin{array}{c}\mathrm{Gr}^W_j\mathrm{CH}^r_{\mathrm{hom}}(X\setminus Y,m)\otimes\mathbb{Q}\\ \downarrow\\ \frac{\mathrm{Ext}^1_{\mathrm{MHS}}\left(\mathbb{Q}(0),\mathrm{Gr}^W_{j-1}H^{2r-m-1}(X\setminus Y,\mathbb{Q}(r))\right)}{\Xi_j},\end{array}$$

for $-m\le \ell\le 0$. *This is evaluated with KLM-formula.*

Remark 2.5.10 Taking only the 0^{th} column, the resulting complex computes the absolute Hodge cohomology of X, $H^{2p+*}(X,\mathbb{A}(p))$. Similarly, each complex column computes the cohomology of $Y_{[i]}$. In the general double complex, if we omit the 0^{th} column, we obtain:

$$H^{BM}_{2r+m}\big(Y,\mathbb{Q}(r)\big)=\mathrm{CH}^r(Y,m)\cong H^{2p-m+2}_{\mathcal{M},Y}\big(X,\mathbb{Q}(r+1)\big)$$

for motivic cohomology with support on Y. In summary, for U smooth and quasi-projective variety and $U\hookrightarrow X$ a smooth compatification of U with $Y:=X-U$ a normal crossing divisor, we get cycle-class morphisms of long exact sequences

$$\begin{array}{ccccc}\mathrm{CH}^{r-1}(Y,m) & \longrightarrow & \mathrm{CH}^r(X,m) & \longrightarrow & \mathrm{CH}^r(U,m)\\ \downarrow & & \downarrow & & \downarrow\\ H^{2r-m}_{\mathcal{H},Y}(X,\mathbb{Q}(r)) & \longrightarrow & H^{2r-m}_{\mathcal{D}}(X,\mathbb{Q}(r)) & \longrightarrow & H^{2r-m}_{\mathcal{H}}(U,\mathbb{Q}(r))\end{array}$$

where both exact sequences are given by the localization sequence on the level of complexes [KL07].

Chapter 3

Motivic cohomology for singular varieties

In [Han00], Hanamura extends the definition of the motive of quasi-projective varieties to schemes of finite type over a field k that admits resolution of singularities, using the method of Hironaka's resolution of singularities [Hir64], and more generally cubical hyperresolutions [GNPP88]. The same technique is used by Levine in [Lev98], to define his cohomological motives. For a quasi-projective variety X, Hanamura applies the technique of hyperresolutions to the cycle theory of S. Bloch and defines its motivic cohomology.

3.1. Cubical hyperresolutions

The technique of cubical hyperresolutions was developed by Guillén-Navarro-Pascual-Puerta in [GNPP88], as an alternative to simplicial resolutions for Deligne theory of mixed Hodge structures [HodgeIII]. In this theory one replaces a singular variety by a cubical diagram of smooth varieties. Here we review the construction of cubical hyperresolutions, and some examples. We recall some notions from [GNPP88] and [GNA02], see also [PS08].

Notation and conventions

From this chapter on, k will be a field that admits resolution of singularities. The category of separated schemes of finite type over k is denoted by $\mathbf{Sch}(k)$. Denote the category of quasi-projective varieties by $\mathbf{QuProj}(k)$, by $\mathbf{Sm}(k)$ to the category of smooth varieties, and by $\mathbf{SmProj}(k)$ the category of smooth and projective varieties over k. Let I be the category associated to a finite partially ordered set. By an **I-diagram of schemes** we mean a contravariant functor $X_\bullet \colon I \to \mathbf{Sch}(k)$, from the category I to the category of schemes. The I-schemes form a category. If $X_\bullet \colon I \to \mathbf{Sch}(k)$ is an I-scheme, and $i \in \mathrm{Ob}(I)$, then X_i will denote the scheme corresponding to i. If $\phi \colon i \to j$ is a morphism in I, will be denote by $X_\phi \colon X_j \to X_i$ the corresponding morphism of schemes. If $f_\bullet \colon X_\bullet \to Y_\bullet$ is a morphism of I-schemes, we denote by f_i the induced morphism $X_i \to Y_i$. We say that a morphism $f_\bullet \colon X_\bullet \to Y_\bullet$ of I-diagrams is a closed inmersion, projective, proper, separated, etc., if $f_i \colon X_i \to Y_i$ is a closed immersion, projective, proper, separated, etc., for all $i \in I$.

Definition 3.1.1 Given a morphism of I-schemes $f_\bullet \colon X_\bullet \to Y_\bullet$, we define the **discriminant** of f to be the smallest closed sub-I-scheme $Z_\bullet$ of $Y_\bullet$ such that $f_i \colon (X_i - f_i^{-1}(Z_i)) \to (Y_i - Z_i)$ is an isomorphism for all $i \in \mathrm{Ob}(I)$.

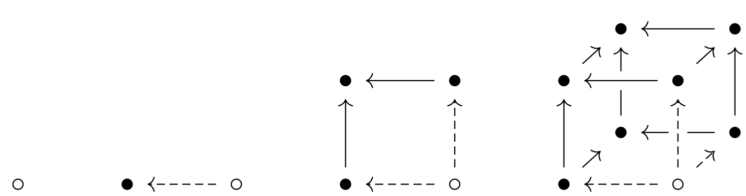
3.1.2 The cubical category. Let $\underline{1}$ be the category $\{0\}$, and $\underline{2}$ be the category $\{0 \to 1\}$ ($\{0 < 1\}$). Let $n \geq 1$ be an integer. We denote by $\square_n^+$ the product of $n+1$ copies of the category $\underline{2} = \{0 \to 1\}$, i.e.,

$$\square_n^+ := (\{0 \to 1\})^{n+1}.$$

The objects of $\square_n^+$ are identified with the sequences $\alpha = (\alpha_0, \ldots, \alpha_n)$ such that $\alpha_i \in \{0, 1\}$ for $0 \leq i \leq n$, and it is useful to define $|\alpha| = \sum \alpha_i$. The morphisms $\mathrm{Hom}_{\square_n^+}(\alpha, \beta)$, for two objects $\alpha, \beta \in \square_n^+$ are given by:

$$\mathrm{Hom}_{\square_n^+}(\alpha, \beta) := \begin{cases} \alpha \to \beta & \text{if } \alpha_i \leq \beta_i \text{ for } 0 \leq i \leq n \\ \emptyset & \text{otherwise.} \end{cases}$$

For $n = -1$, we set $\square_{-1}^+ = \{0\}$ and for $n = 0$ we have $\square_0^+ = \{0 \to 1\}$. Let $\square_n$ be the full subcategory without the initial object $(0, \ldots, 0)$. Clearly, the category $\square_n^+$ can be identified with the category $\square_n$ with an augmentation morphism to $\{0\}$. The objects in the category $\square_n^+$ can be seen as:



for $n = \{-1, 0, 1, 2\}$. The category $\square_n$ are the same diagrams without the initial object ($\square_n = \square_n^+ - \{(0, \ldots, 0)\}$).

Definition 3.1.3 An **augmented cubical variety** or $\square_n^+$**- variety** (resp. a cubical variety or $\square_n$-variety) on $\mathbf{QuProj}(k)$ is defined as a contravariant functor $X_\bullet^+ \colon \square_{n-1}^+ \to \mathbf{QuProj}(k)$ (resp. $X_\bullet \colon \square_{n-1} \to \mathbf{QuProj}(k)$).

Morphisms between two cubical varieties are given by natural transformations, and then cubical varieties form a category. In order to construct cubical hyperresolutions, we start with the notion of a 2-resolution, which is an iterative step for such a construction.

Definition 3.1.4 ([GNPP88, I.2.7]) Let X be a I-scheme. The Cartesian square (up to taking the reduced scheme structure, i.e., $X_{11} = (X_{10} \times_X X_{01})_{\mathrm{red}}$) of morphism of I-schemes

$$\begin{tikzcd} X_{11} \arrow[r] \arrow[d] & X_{01} \arrow[d, "f"] \\ X_{10} \arrow[r] & X \end{tikzcd}$$

is called a **2-resolution** of X if it satisfies the following conditions:

(1) X_{01} is a smooth I-scheme,

(2) the horizontal arrows are closed inmersions of I-schemes,

(3) f is a proper I-morphism,

(4) X_{10} contains the discriminant of f.

In other words, the condition (4) says that f induces an isomorphism of $(X_{01})_i - (X_{11})_i$ over $(X)_i - (X_{10})_i$ for all $i \in \mathrm{Ob}(I)$. Clearly 2-resolutions always exist under the same hypotheses that resolutions of I-schemes exist.

3.1.5 The reduction process. The previous 2-resolution, $X^1_\bullet \to X$, can be seen as a morphism of $\square_0^+$-varieties $X_{1\bullet} \to X_{0\bullet}$, where $X_{00} = X$. For a 2-resolution $X^2_\bullet$ of $X_{1\bullet}$, we can define a $\square_2^+$-variety by reducing the two 2-resolutions $X^2_\bullet \to X^1_\bullet \to X$ as

$$\begin{tikzcd} X^2_{11\bullet} \arrow[r] \arrow[d] & X^2_{01\bullet} \arrow[d] \\ X^2_{10\bullet} \arrow[r] & X^1_{0\bullet} \end{tikzcd}$$

in other words, the diagram suppresses the vertex $X^2_{00\bullet} = X^1_{1\bullet}$. This construction motivates the following definition, important for the notion of cubical hyperresolution.

Definition 3.1.6 ([GNPP88, I.2.11]) Let $r \geq 1$ be an integer, and $X_\bullet^n$ be a $\square_n^+ \times I$-scheme for $1 \leq n \leq r$. Suppose that for all $1 \leq n \leq r$, the $\square_{n-1}^+ \times I$-schemes $X_{00\bullet}^{n+1}$ and $X_{1\bullet}^n$ are equal. Then we define, by induction on r, the $\square_r^+ \times I$-scheme

$$Z_\bullet := \mathrm{red}(X_\bullet^1, X_\bullet^2, \ldots, X_\bullet^r)$$

which we call the **reduction** of $(X_\bullet^1, X_\bullet^2, \ldots, X_\bullet^r)$, in the following way:

(1) If $r = 1$, we define $Z_\bullet = X_\bullet^1$.

(2) if $r = 2$, we define $Z_\bullet = \mathrm{red}(X_\bullet^1, X_\bullet^2)$ by

$$Z_{\alpha\beta} = \begin{cases} X_{0\beta}^1 & \text{if} \quad \alpha = (0,0) \\ X_{\alpha\beta}^2 & \text{if} \quad \alpha \in \square_1 \end{cases}$$

for all $\beta \in \square_0^+$, with the obvious morphisms.

(3) If $r > 2$, we define recursively

$$Z_\bullet = \mathrm{red}\big(\mathrm{red}(X_\bullet^1, \ldots, X_\bullet^{r-1}), X_\bullet^r\big).$$

Now we come to the definition of a cubical hyperresolution, the principal definition of this section.

Definition 3.1.7 ([GNPP88, I.2.12]) Let X be an I-scheme. A $\square_r^+ \times I$-scheme $Z_\bullet = \mathrm{red}(X_\bullet^1, X_\bullet^2, \ldots, X_\bullet^r)$ is called an (augmented) **cubical hyperresolution** of X if

(1) $X_\bullet^1$ is a 2-resolution of X,

(2) for $1 \leq n < r$, $X_\bullet^{n+1}$ is a 2-resolution of $X_\bullet^n$,

(3) X_α is smooth for all $\alpha \in \square_r$.

The number r is the **length** of hyperresolution $X_\bullet$. The machinery behind cubical hyperresolutions, is an iterative process of 2-resolutions with its corresponding reductions.

Notation. A cubical hyperresolution $X_\bullet = \mathrm{red}(X_\bullet^1, X_\bullet^2, \ldots, X_\bullet^r)$ naturally defines a semi-simplicial scheme[1], where the p-th component is

$$X_p = \coprod_{|\alpha|=p+1} X_\alpha, \quad \text{with} \qquad |\alpha| = |(\alpha_0, \ldots, \alpha_m)| = \alpha_0 + \cdots + \alpha_m;$$

[1] Also called *strict simplicial scheme*, in our case it is also *truncated*.

we denote this by the same $X_\bullet$. A **semi-simplicial scheme** is a sequence of varieties X_p together with face morphisms $d_i\colon X_p \to X_{p-1}$ for $0 \le i \le p$, and augmentation morphism a over X:

$$\cdots \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow}\; X_2 \;\substack{\xrightarrow{d_2}\\ \xrightarrow{d_1}\\ \xrightarrow[d_0]{}}\; X_1 \;\substack{\xrightarrow{d_1}\\ \xrightarrow[d_0]{}}\; X_0 \xrightarrow{a} X$$

such that $d_i \circ d_j = d_{j-1} \circ d_i$ for $0 \le i < j \le p$, and $a \circ d_0 = a \circ d_1$. Contrary to the simplicial schemes, here the degeneration morphisms are irrelevant for our following purposes. A **morphism of semi-simplicial schemes** $f_\bullet\colon X_\bullet \to X'_\bullet$ is a family of morphisms $\{f_p\colon X_p \to X'_p\}$ such that all the f_p's commute with the face morphisms of $X_\bullet$ and $X'_\bullet$. We denote an augmented hyperresolution $X_\bullet^+$ as $a\colon X_\bullet \to X$, such that X_p is smooth for all p, and $a_p\colon X_p \to X$ are proper. In this sense, $X_\bullet \to X$ can be seen as a morphism of semi-simplicial schemes, by considering X as simplicial scheme with $X_p = X$ for all $p \in \mathbb{Z}_{\ge 0}$ and by setting all face morphisms to be the identity morphism.

Construction and existence of cubical hyperresolutions

Definition 3.1.8 Let k be a field, and X be a quasi-projective variety over k with singular locus Σ. A **resolution of singularities** of X is a proper birational morphism $p\colon \widetilde{X} \to X$, where $\widetilde{X}$ is a smooth variety; p induces an isomorphism outside Σ and $E = p^{-1}(\Sigma)$ is a simple normal crossing divisor.

In characteristic zero, Hironaka [Hir64] proves that there is resolution of singularities for any variety. Futhermore, these resolutions are compositions

$$\widetilde{X} \longrightarrow X_n \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 = X$$

of finetely many blow-ups along the smooth centers $Z_i \subset X_i$.

Under the current assumption that resolutions of singularities exist, a cubical hyperresolution of any variety exists. Cubical hyperresolution is an iterative process of Hironaka's resolution of singularities[2]. For proofs about the existence of cubical hyperresolutions, see [GNPP88, I.2.15] and [PS08, Theo. 5.26]. For our future purposes, the following theorem is enough.

[2] In prime charactiristic, alterations in the sense of de Jong give another way to construct hyperresolutions.

Theorem 3.1.9 *Let X be a quasi-projctive variety over a field k of characteristic 0. Then there exists a cubical hyperresolution $X_\bullet$ of X such that* $\dim X_\alpha \leq \dim X - |\alpha| + 1$*, for all* $\alpha \in \square_n^+$.

Proof. The first step in the recurrence is the resolution of singularities of X, this is a diagram of the form

$$\begin{tikzcd} E \arrow[r, "i'"] \arrow[d, "p'"'] & \widetilde{X} \arrow[d, "p"] \\ \Sigma \arrow[r, "i"'] & X \end{tikzcd}$$

This diagram defines is so-called a 2-resolution $X_\bullet^1 \to X$ of X

$$\begin{tikzcd} X_{11}^1 \arrow[r] \arrow[d] & X_{01}^1 \arrow[d, "p"] \\ X_{10}^1 \arrow[r] & X_{00} = X \end{tikzcd}$$

If X_{11}^1 and X_{10}^1 are smooth, this 2-resolution $X_\bullet^1 \to X$ defines an hyperresolution of X with associated semi-simplicial scheme given by

$$X_1 = X_{11} \rightrightarrows X_0 = X_{10} \coprod X_{01} \longrightarrow X$$

with an augmentation over X (of length one). If X_{11}^1 and X_{10}^1 are not smooth, since $\dim(X_{11}^1), \dim(X_{10}^1) < \dim(X)$, the inductive process continue applying again resolution of singularities on X_{11}^1 and X_{10}^1 to construct a 2-resolution of $(X_{11}^1 \to X_{10}^1) = X_{1\bullet}^1$, this is $X_\bullet^2 \to X_\bullet^1 \to X$. The reduction process eliminates the non-smooth components $X_{00\bullet}^2 = X_{1\bullet}^1$, and replaces them with the components $X_{0\bullet}^1$

$$\begin{tikzcd}[column sep=small] & X_{111}^2 \arrow[rr] \arrow[dl] \arrow[dd] & & X_{011}^2 \arrow[dl] \arrow[dd] \arrow[ddrrr] & & & \\ X_{110}^2 \arrow[rr] \arrow[dd] & & X_{010}^2 \arrow[dd] \arrow[ddrrr] & & & & \\ & X_{101}^2 \arrow[rr] \arrow[dl] & & X_{001}^2 \arrow[r, equal] \arrow[dl] & X_{11}^1 \arrow[rr] \arrow[dl] & & X_{01}^1 \arrow[dl] \\ X_{100}^2 \arrow[rr] & & X_{000}^2 \arrow[r, equal] & X_{10}^1 \arrow[rr] & & X_{00}^1 & \end{tikzcd}$$

If all the other vertices X_α are smooth, this gives a hyperresolution of X. Again, $\dim X_\alpha \leq \dim X - |\alpha| + 1$ for all $\alpha \in \square_2^+$. Again, this cube gives rise to a semi-simplicial scheme of the form

$$X_2 = X_{111} \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} X_1 = X_{110} \coprod X_{101} \coprod X_{011} \rightrightarrows X_0 = X_{100} \coprod X_{010} \coprod X_{001}$$

with an augmentation over X. If not all components are smooth, we proceed in the same way to take a 2-resolution $X^3_\bullet \to (X^2_{11\bullet} \to X^2_{10\bullet})$ of one face of the cube:

$$\begin{array}{ccc} X^2_{111} & \longrightarrow & X^2_{110} \\ \downarrow & & \downarrow \\ X^2_{101} & \longrightarrow & X^2_{100} \end{array}$$

and so on, this process is bounded by the dimension of X. At the end, after a finite number of steps, we get to that all the vertices of the r-dimensional cube are smooth, then we obtain a cubical hyperresolution (under reductions) $X_\bullet = \mathrm{red}(X^1_\bullet, X^2_\bullet, \dots, X^r_\bullet) \to X$, with an augmented semi-simplicial scheme $X_\bullet \to X$. □

Remark 3.1.10 The main techinical advantage of Guillén, Navarro-Aznar et al. constructions over Deligne's construction, is that resulting semi-simplicial scheme is finite, bounded by the dimension of X.

Example 3.1.11 Let C be an algebraic curve over k. Consider the normalization $\eta\colon \widetilde{C} \to C$ and the singular locus $\Sigma \subset C$. In particular, Σ is the discriminant of η and $E = \eta^{-1}(\Sigma)$ the exceptional set (with reduced scheme structure). We have the following Cartesian square:

$$\begin{array}{ccc} E & \longrightarrow & \widetilde{C} \\ \downarrow & & \downarrow{\scriptstyle \eta} \\ \Sigma & \longrightarrow & C. \end{array}$$

It is clearly a 2-resolution and thus also a cubical hyperresolution of C.

Example 3.1.12 Let Y be a **connected normal crossing variety**, i.e., $Y = \bigcup_{i=1}^N Y_i$, where each irreducible component is smooth and projective. Set

$$Y_I := Y_{i_1} \cap \cdots \cap Y_{i_t}, \quad I = \{1, \dots, t\}$$
$$Y_{[t]} = \coprod_{|I|=t} Y_I.$$

For $I = \{i_1, \dots, i_t\}$, consider the following notation $I_j = \{i_1, \dots, \widehat{i_j}, \dots, i_t\}$; therefore, there exist t natural inclusions $d^j_I\colon E_I \to E_{I_j}$. Then, the inclusions define a semi-simplicial hyperresolution $Y_{[\bullet]} \to Y$ of Y, given by:

$$\cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} Y_{[3]} \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} Y_{[2]} \substack{\longrightarrow\\\longrightarrow} Y_{[1]} \xrightarrow{a} Y.$$

A complex on $Y_{[\bullet]}$ is a double complex, with a complex over each $Y_{[t]}$ compatible with face morphisms. If a is of cohomological descent, the simple complex associated to the double complex computes the cohomology of Y. In this case, the spectral sequence associated to hyperresolution is the Mayer-Vietoris (or Čech) spectral sequence associated to decomposition. This is the principal role plays the descent cohomological property (see 4.2.1).

Theorem 3.1.13 ([GNPP88, I.6.9]) *Let X be an I-scheme. If $X_\bullet \to X$ is a cubical hyperresolution of X, then $X_\bullet$ has cohomological descent over X.*

3.1.14 The category of hyperresolutions. A *morphism of hyperresolutions* over a morphism $f\colon X \to X'$ is given by taking an inclusion functor $i_{r,r'} = i_1 \circ \cdots \circ i_{r'-r}\colon \square_r^+ \to \square_{r'}^+$ $(r \le r')$, where $i_j\colon \square_r^+ \to \square_{r+1}^+$ sends $(\alpha_0, \ldots, \alpha_r)$ to $(\alpha_0, \ldots, \alpha_{j-1}, 0, \alpha_j, \cdots, \alpha_r)$, and taking a morphism of $\square_r^+ \times I$-diagrams $f_\bullet\colon X_\bullet \to X'_\bullet$ such that the morphism $f_{(0,\ldots,0)}$ is f. Cubical hyperresolutions of I-schemes form a category. A morphism of hyperresolutions induces the associated morphism of augmented semi-simplicial schemes, also denoted by $f_\bullet\colon X_\bullet \to X'_\bullet$. Denote by **Hrc** the category of hyperresolutions of quasi-projective 1-varieties over k. The above construction gives us a natural functor

$$\begin{array}{rcl} \mathbf{Hrc} & \to & \mathbf{QuProj}(k) \\ X_\bullet & \mapsto & X. \end{array}$$

by taking the $(0, \ldots, 0)$ component of the associated augmented hyperresolution. Localizing **Hrc** with respect to the class Σ of morphisms which induces identities in $\mathbf{QuProj}(k)$, we obtain the category **HoHrc**. Hence, we have an induced functor

$$\mathbf{HoHrc} \to \mathbf{QuProj}(k).$$

This functor is an equivalence of categories [GNPP88, Theo. 3.8].

3.2. Chow cohomology groups

Let X be a quasi-projective variety over $\mathbb{C}$. The principal idea of cubical hyperresolutions is to replace a singular variety X by a semi-simplicial variety $a\colon X_\bullet \to X$ [GNPP88]. This is a truncated semi-simplicial scheme

$$X_\bullet = \left\{ \cdots \; \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow} \; X_2 \; \substack{\xrightarrow{d_0} \\ \xrightarrow{d_1} \\ \xrightarrow[d_2]{}} \; X_1 \; \substack{\xrightarrow{d_0} \\ \xrightarrow[d_1]{}} \; X_0 \right\} \xrightarrow{a} X$$

consisting of smooth quasi-projective varieties, with an augmentation a to X. In [Han00, Theo. 2.3], Hanamura proves that the homological cycle complex has the property of descent, in other words, the morphism of complexes $a_* \colon \mathcal{Z}_r(X_\bullet)_* \to \mathcal{Z}_r(X, \bullet)$ is a quasi-isomorphism, this is a consequence of the localization theorem of Bloch's cycle complexes. Here, $\mathcal{Z}_r(X_\bullet)_*$ is the double complex with differentials $d_* = \sum(-1)^i d_{i*}$. On the cohomological side, we have the following construction.

3.2.1 Hanamura's motivic cohomology. Let X be a quasi-projective variety, and $X_\bullet \to X$ its semi-simplicial hyperresolution. For each X_p take its cycle complex $\mathcal{Z}^r(X_p, \bullet)$, and form a double complex

$$0 \longrightarrow \mathcal{Z}^r(X_0, \bullet) \xrightarrow{d^*} \mathcal{Z}^r(X_1, \bullet) \xrightarrow{d^*} \cdots \xrightarrow{d^*} \mathcal{Z}^r(X_\alpha, \bullet) \xrightarrow{d^*} \cdots$$

where the horizontal differentials are given by $d^* = \sum(-1)^i d_i^*$, the alternating sums of the pull-backs of face maps. We denote by $\mathcal{Z}^r(X_\bullet)^*$ its total complex and we call it the **cohomological cycle complex** of X.

Remark 3.2.2 In Hanamura's construction, strictly speaking one chooses appropiate distinguished subcomplexes (quasi-isomorphic to Bloch's cycle complexes), so that the pull-back d^* is well-defined [Han00]. For smooth, quasi-projective varieties X, the collection of distinguished subcomplexes satisfies the following conditions:

- The inclusion of a distinguished subcomplex $\mathcal{Z}^r(X, \bullet)' \subset \mathcal{Z}^r(X, \bullet)$ is a quasi-isomorphism.
- If $\mathcal{Z}^r(X, \bullet)'$ and $\mathcal{Z}^r(X, \bullet)''$ are distinguished subcomplexes of $\mathcal{Z}^r(X, \bullet)$, there is a third distinguished subcomplex $\mathcal{Z}^r(X, \bullet)'''$ contained in both $\mathcal{Z}^r(X, \bullet)'$ and $\mathcal{Z}^r(X, \bullet)''$.
- If $f \colon X \to Y$ is a morphism of smooth, quasi-projective varieties, and $\mathcal{Z}^r(X, \bullet)' \subset \mathcal{Z}^r(X, \bullet)$, there is a distinguished subcomplex $\mathcal{Z}^r(Y, \bullet)' \subset \mathcal{Z}^r(Y, \bullet)$ such that the morphism $f^* \colon \mathcal{Z}^r(Y, \bullet)' \to \mathcal{Z}^r(X, \bullet)'$ is well defined.

The main result on the cohomological cycle complex is that it is independent of the hyperresolution up to isomorphisms in the derived category.

Theorem 3.2.3 ([Han00, Theorem I]) *Let X be a quasi-projective variety, $X_\bullet \to X$ and $X'_\bullet \to X$ be hyperresolutions of X, and $X_\bullet \to X'_\bullet$ a* *morphism over X. Then the induced morphism $\mathcal{Z}^r(X'_\bullet)^* \to \mathcal{Z}^r(X_\bullet)^*$ is a quasi-isomorphism. There is a contravariant functor*

$$\mathcal{Z}^r(-)^* \colon \mathbf{QuProj}(k) \to \mathbf{D}^+(\mathbb{Q})$$

that sends X to $\mathcal{Z}^r(X_\bullet)^*$ and a morphism $f\colon X \to Z$ the induced morphism $(f_\bullet)^*\colon \mathcal{Z}^r(X_\bullet)^* \to \mathcal{Z}^r(Z_\bullet)^*$, where $f_\bullet\colon X_\bullet \to Z_\bullet$ is a morphism of semi-simplicial hyperresolutions over f.

In terms of motives, Levine [Lev98, IV.3] associates to a hyperresolution $X_\bullet$ of X a motive $\mathbb{Z}_{X_\bullet}$ in $\mathbf{D}^b_{\mathrm{mot}}(\mathbf{Sm}(k))$. The morphism of hyperresolutions gives a morphism of motives $(f_\bullet)^*\colon \mathbb{Z}_{X_\bullet} \to \mathbb{Z}_{X'_\bullet}$ in $\mathbf{C}^b_{\mathrm{mot}}((\mathbf{Sm}(k))$, canonically isomorphic in $\mathbf{D}^b_{\mathrm{mot}}(\mathbf{Sm}(k))$. In order to prove this and Theorem 3.2.3, Hanamura considers a Cartesian square of quasi-projective varieties

$$\begin{tikzcd} W' \arrow[r, "\alpha"] \arrow[d, "g"] & X' \arrow[d, "f"] \\ W \arrow[r, "\beta"] & X \end{tikzcd}$$

such that f is proper and induces an isomorphism $(X' - W') \to (X - W)$, with corresponding diagram of hyperresolutions

$$\begin{tikzcd} W'_\bullet \arrow[r, "\alpha_\bullet"] \arrow[d, "g_\bullet"] & X'_\bullet \arrow[d, "f_\bullet"] \\ W_\bullet \arrow[r, "\beta_\bullet"] & X_\bullet \end{tikzcd}$$

over the above square. Then, the induced morphism

$$(f^*_\bullet, g^*_\bullet)\colon \mathrm{Cone}(\beta^*_\bullet) \to \mathrm{Cone}(\alpha^*_\bullet)$$

is an isomorphism in $\mathbf{D}^b_{\mathrm{mot}}(\mathbf{Sm}(k))$ (respectively in $\mathbf{D}^b(\mathbb{Q})$). Once we have the independence of hyperresolution, we give the following definition:

Definition 3.2.4 For a quasi-projective variety X, we define

$$\mathrm{CHC}^r(X, m) := H_m\big(\mathcal{Z}^r(X_\bullet)^*\big)$$

called **higher Chow cohomology group**[3] of X.

[3]Or **motivic cohomology** of X in Hanamura's sense of triangulated mixed motives.

3.2.5 Hanamura's spectral sequence. For a quasi-projective variety X, consider its semi-simplicial hyperresolution $X_\bullet \to X$. The cohomological cycle complex $\mathcal{Z}^r(X_\bullet)^*$ of X defines a fourth quadrant double complex:

$$E_0^{p,q}(r) := \mathcal{Z}^r(X_p, -q); \quad i \geq 1, j \leq 0.$$

This complex can be seen as

$$\begin{array}{ccccccc}
\mathcal{Z}^r(X_0) & \xrightarrow{d^*} & \mathcal{Z}^r(X_1) & \xrightarrow{d^*} & \mathcal{Z}^r(X_2) & \longrightarrow & \\
\partial\Big\uparrow & & \partial\Big\uparrow & & \partial\Big\uparrow & & \\
\mathcal{Z}^r(X_0,1) & \xrightarrow{d^*} & \mathcal{Z}^r(X_1,1) & \xrightarrow{d^*} & \mathcal{Z}^r(X_2,1) & \longrightarrow & \\
\partial\Big\uparrow & & \partial\Big\uparrow & & \partial\Big\uparrow & & \\
\mathcal{Z}^r(X_0,2) & \xrightarrow{d^*} & \mathcal{Z}^r(X_1,2) & \xrightarrow{d^*} & \mathcal{Z}^r(X_2,2) & \longrightarrow & \\
\Big\uparrow & & \Big\uparrow & & \Big\uparrow & &
\end{array}$$

whose differentials are ∂ vertically (∂ as coming from the definition of Bloch's cycle complex) and d^* are the alternating sums of the pull-backs. Again, denote the total complex by $\mathcal{Z}^r(X_\bullet, \cdot)^*$.

Proposition 3.2.6 *There is a cohomological spectral sequence associated to the Chow cohomology double complex*

$$E_1^{p,q}(r) := \mathrm{CH}^r(X_p, -q) \Rightarrow \mathrm{CHC}^r(X, p-q).$$

In the homological sense, there is a first quadrant convergent spectral sequence

$$E^1_{p,q}(r) := \mathrm{CH}_r(X_p, q) \Rightarrow \mathrm{CH}_r(X, p+q).$$

Proof. Let $X_\bullet \to X$ be the augmented semi-simplicial scheme given by a cubical hyperresolution of X. The first spectral sequence is trivial from the definition of Chow cohomology groups. The second one is a consequence of descent property for cycle complexes, the complex $\mathcal{Z}_r(X_\bullet)_*$ is quasi-isomorphic $\mathcal{Z}_r(X, \bullet)$ and we have a first quadrant spectral sequence $E^1_{p,q}(r) := \mathrm{CH}_r(X_p, q) \Rightarrow \mathrm{CH}_r(X, p+q)$. $\square$

Theorem 3.2.7 *The Chow cohomology groups have the following properties:*

(1) *The assignament* $X \mapsto \mathrm{CHC}^r(X,m)$ *defines a contravariant functor; a map* $f\colon X \to Y$ *induces a morphism* $f^*\colon \mathrm{CHC}^r(Y,m) \to \mathrm{CHC}^r(X,m)$.

(2) *If* X *is smooth, then* $\mathrm{CHC}^r(X,m) = \mathrm{CH}^r(X,m)$.

(3) **Homotopy invariance:** *The projection* $X \times \mathbb{A}^1 \to X$ *induces an isomorphism*
$$\mathrm{CHC}^r(X,m) \cong \mathrm{CHC}^r(X \times \mathbb{A}^1, m).$$

(4) **Mayer-Vietoris:** *For an open covering* $X = U \cup V$*:*
$$\to \mathrm{CHC}^r(X,m) \to \mathrm{CHC}^r(U,m) \oplus \mathrm{CHC}^r(V,m) \to \mathrm{CHC}^r(U \cap V, m) \to$$

Proof. The first two properties are clear. The homotopy property is given by the homotopy property of Bloch's higher Chow groups (for smooth varieties), together with the fact that if $X_\bullet$ is a semi-simplicial hyperresolution of X, then $X_\bullet \times \mathbb{A}^1$ is a semi-simplicial hyperresolution of $X \times \mathbb{A}^1$. Via Hanamura's spectral sequence

$$\begin{tikzcd}
E_1^{p,q}(r) = \mathrm{CH}^r(X_p, -q) \arrow[r, Rightarrow] \arrow[d, "\cong"] & \mathrm{CHC}^r(X, p-q) \arrow[d] \\
E_1^{p,q}(r) = \mathrm{CH}^r(X_p \times \mathbb{A}^1, -q) \arrow[r, Rightarrow] & \mathrm{CHC}^r(X \times \mathbb{A}^1, p-q)
\end{tikzcd}$$

we have the isomorphism $\mathrm{CHC}^r(X, p-q) \cong \mathrm{CHC}^r(X \times \mathbb{A}^1, p-q)$. For the Mayer-Vietoris property, again we consider a cubical hyperresolution $X_\bullet \to X$ of X. Then the base change $U_\bullet := X_\bullet \times_X U$, $V_\bullet := X_\bullet \times_X V$, and $(U \cap V)_\bullet := X_\bullet \times_X (U \cap V)$:

$$\begin{tikzcd}
X_\bullet \times_X U \arrow[r] \arrow[d] & X \arrow[d] & X_\bullet \times_X V \arrow[r] \arrow[d] & X \arrow[d] & X_\bullet \times_X (U \cap V) \arrow[r] \arrow[d] & X \arrow[d] \\
U \arrow[r] & X & V \arrow[r] & X & (U \cap V) \arrow[r] & X
\end{tikzcd}$$

form hyperresolutions of U, V, and $(U \cap V)$ respectively. Thus we obtain the following diagram:

$$\begin{tikzcd}
(U \cap V)_\bullet \arrow[r] \arrow[d] & U_\bullet \arrow[d] \\
V_\bullet \arrow[r] & X_\bullet
\end{tikzcd}$$

with $U_\alpha \cup V_\alpha = X_\alpha$ and $U_\alpha \cap V_\alpha = (U \cap V)_\alpha$. Then the morphisms

$$\mathcal{Z}^r(X_\alpha, \bullet) \to \mathrm{Cone}^\bullet[\mathcal{Z}^r(U_\alpha, \bullet) \oplus \mathcal{Z}^r(V_\alpha, \bullet) \to \mathcal{Z}^r(U_\alpha \cap V_\alpha, \bullet)]$$

are quasi-isomorphis for all $\alpha \in \square_r$, this leads to $\mathcal{Z}^r(X_\bullet)^*$ and induces the long exact Mayer-Vietoris sequence. $\square$

Example 3.2.8 ([Han14, Prop. 1.1, 1.2]) Let C be a quasi-projective curve over k. Let $\eta\colon \widetilde{C} \to C$ be the normalization of C, with $\Sigma \subset C$ the singular locus and $E = \eta^{-1}(\Sigma)$. In this case η is a resolution of singularities and defines a commutative square, that gives a cubical hyperresolution of C (3.1.11). Taking the semi-simplicial hyperresolution, we have by definition

$$\mathcal{Z}^r(C, \bullet)^* := \mathrm{Cone}\{\mathcal{Z}^r(\widetilde{C}, \bullet) \oplus \mathcal{Z}^r(\Sigma, \bullet) \to \mathcal{Z}^r(E, \bullet)\}[-1].$$

Then, there is a long exact sequence

$$\begin{aligned}\cdots \to \mathrm{CHC}^r(C, m) \to \mathrm{CH}^r(\widetilde{C}, m) \oplus \mathrm{CH}^r(\Sigma, m) \to \mathrm{CH}^r(E, m)\\ \to \mathrm{CHC}^r(C, m-1) \to \cdots\end{aligned}$$

By (2.2.4), if $r = 1$ and X is smooth, then $\mathrm{CH}^1(X, m) = 0$ for $m \neq 0, 1$ and $\mathrm{CH}^1(X, 1) = \Gamma(X, \mathcal{O}_X^*)$. Then $\mathrm{CHC}^1(C, m) = 0$ for $m \neq 0, 1$, and we have the exact sequence

$$\begin{aligned}0 \to \mathrm{CHC}^1(C, 1) \to \Gamma(\widetilde{C}, \mathcal{O}_{\widetilde{C}}^*) \oplus \Gamma(\Sigma, \mathcal{O}_\Sigma^*) \to \Gamma(E, \mathcal{O}_E^*)\\ \to \mathrm{CHC}^1(C) \to \mathrm{CH}^1(\widetilde{C}) \to 0.\end{aligned}$$

Now, $\Gamma(\widetilde{C}, \mathcal{O}_{\widetilde{C}}^*) = k^*$, $\Gamma(\Sigma, \mathcal{O}_\Sigma^*) = \bigoplus_{p \in \Sigma} k^*$ and $\Gamma(E, \mathcal{O}_E^*) = \bigoplus_{q \in E} k^*$. If C is irreducible and projective, then $\mathrm{CHC}^1(C, 1) = k^*$ and there is an exact sequence:

$$0 \to \bigoplus_{p \in \Sigma} \big(\bigoplus_{q \mapsto p} k^*\big)/k^* \to \mathrm{CHC}^1(C) \to \mathrm{CH}^1(\widetilde{C}) \to 0.$$

Theorem 3.2.9 ([Han14, Theorem 3.3]) *Let S be an irreducible normal quasi-projective surface over an algebracally closed field of characteristic zero, and E the execptional divisor given by a desingularization of S. Then the canonical morphism*

$$\mathrm{CHC}^r(S, n)_\mathbb{Q} \to \mathrm{CH}_{2-r}(S, n)_\mathbb{Q}$$

is an isomorphism for all r, n if only if E is a rational tree.

Remark 3.2.10 Hanamura's construction can be seen as an application of the principle of extension of Guillén-Navarro [GNA02], applied to higher Chow groups, then many of the properties for smooth varieties can be inherited to singular varieties.

3.3. Voevodsky's motivic cohomology

The purpose of this section is to see the relationship between Hanamura's Chow cohomology groups [Han00] and the motivic cohomology of Friedlander-Voevodsky [FV00]. This result is a consequence of the quasi-isomorphism between Bloch's cycle complexes and Friedlander-Suslin complexes in the smooth case [MVW06, Theo. 19.8], using the property of homotopy invariance and spectral sequences. The motivic cohomology for a singular variety is defined in terms of the cdh-cohomology. For this purpose we need to introduce the cdh-topology, which permits to consider blow-up exact sequences [MVW06], [FS02] and [VSF00].

The cdh-topology

Recall the definition of the *Nisnevich topology* [FV00]. A family of étale morphisms $\{p_i \colon U_i \to X\}$ is called a **Nisnevich covering** of X if it has the Nisnevich lifting covering: if for every point $x \in X$ there is an index i and a point $u \in U_i$ with $p_i(u) = x$ such that the induced map of residue fields $k(x) \to k(u)$ is an isomorpism. Nisnevich coverings satisfy the axioms for a Grothendieck topology, which is called the **Nisnevich topology** in $\mathbf{Sch}(k)$. This topology is intermediate between étale topology and Zarsiki topology. To apply resolution of singularities, Voevodsky introduced the *cdh-topology.*

Definition 3.3.1 An **abstract blow-up square** is a diagram of pullback

$$\begin{array}{ccc} E & \longrightarrow & X' \\ \downarrow & & \downarrow{\scriptstyle p} \\ \Sigma & \xrightarrow{i} & X \end{array}$$

with $i \colon \Sigma \hookrightarrow X$ a closed embedding and $p \colon \widetilde{X} \to X$ is a proper morphism that induces an isomorphism $(X' - E) \to (X - \Sigma)$. The associated morphism $X' \coprod \Sigma \to X$ is then called the **abstract blow-up cover**.

Definition 3.3.2 The **cdh-topology** (completely decomposed h-topology)[4] on $\mathbf{Sch}(k)$ is the minimal Grothendieck topology generated by Nisnevich covers and covers $X' \coprod \Sigma \to X$ corresponding to abstract blow-ups.

[4] "Completly decomposed" is the original term for Nisnevich topology, and the h-topology was introduced in Voevodsky's thesis.

Example 3.3.3 Resolution of singularities is a form to generate cdh-coverings.

In order to work with singular varieties, Voevodsky defines a complex associated to any separated scheme of finite type, to consider motivic cohomology by taking cdh-cohomology. We introduce the notion of an equidimensional cycles to construct the Suslin-Friedlander chain complex $\mathbb{Z}^{SF}(i)$.

Definition 3.3.4 Let T be a scheme of finite type over k, and $r \geq 0$ an integer. The presheaf $\mathcal{Z}_{\text{equi}}(T, r)\colon \mathbf{Sm}(k)^{\text{op}} \to \mathbf{Ab}$ is defined as follows: for U smooth, $\mathcal{Z}_{\text{equi}}(T, r)(U)$ is the free abelian group generated by the closed, irreducible subvarieties Z of $U \times T$ which are dominant and equidimensional of relative dimension r over a component of U.

The presheaf $\mathcal{Z}_{\text{equi}}(T, r)$ admits an extension to $\mathbf{Corr}_{\text{fin}}(k)$, as a presheaf with transfers. Moreover, $\mathcal{Z}_{\text{equi}}(T, r)$ is a Zariski sheaf, and even étale sheaf, for each T and $r \geq 0$ [MVW06, Lect. 16, pag. 125]. If k admits resolution of singularities, the **Suslin-Friedlander motivic complexes** $\mathbb{Z}^{SF}(i)$ are given by

$$\mathbb{Z}^{SF}(i) := C_* \mathcal{Z}_{\text{equi}}(\mathbb{A}^i, 0)[-2i].$$

We regard $\mathbb{Z}^{SF}(i)$ as a bounded above cochain complex of abelian groups of presheaves associated to the simplicial presheaf on $\mathbf{Sm}(k)$, given by $C_n \mathcal{Z}_{\text{equi}}(T, r)(U) = \mathcal{Z}_{\text{equi}}(T, r)(U \times \Delta^n)$. Recall that **Voevodsky's motivic complex** [MVW06, Def. 3.1] is defined as the complex of presheaf with transfers

$$\mathbb{Z}(q) = C_*\big(\mathbb{Z}_{\text{tr}}(\mathbb{G}_m^{\wedge q})\big)[-q].$$

These complexes are actually complexes of sheaves with respect to the Zariski topology [MVW06, Lemma 3.2]. The **motivic cohomology** of a smooth scheme X is the hypercohomology of the motivic complex $\mathbb{Z}(q)$ in the Zariski topology $H^p_{\mathcal{M}}(X, \mathbb{Z}(q)) := \mathbb{H}^q_{\text{Zar}}(X, \mathbb{Z}(q))$, see [MVW06, Def. 3.4].

Theorem 3.3.5 [MVW06, Theo. 16.7] *There is a quasi-isomorphism between Voevodsky's and Suslin-Friedlander's complexes in the Zariski topology:*

$$\mathbb{Z}(q) \xrightarrow{\text{qis}} \mathbb{Z}^{SF}(q).$$

In particular, we have an isomorphism $\mathbb{H}^q_{\text{Zar}}(X, \mathbb{Z}(q)) \cong \mathbb{H}^q_{\text{Zar}}(X, \mathbb{Z}^{SF}(q))$.

To incorporate resolution of singularities, motivic cohomology for singular varieties is given in terms of the cdh-topology [FV00, Def. 9.2]:

Definition 3.3.6 For any scheme of finite type X over a field k, the **motivic cohomology** of X is defined by

$$H^q_{\mathcal{M}}(X,\mathbb{Z}(j)) := \mathbb{H}^q_{\mathrm{cdh}}\big(X,\mathbb{Z}^{SF}(j)_{\mathrm{cdh}}\big).$$

It is important to consider this definition of motivic cohomology, because the cdh-descent implies that for a 2-resolution there exists a long exact sequence [FV00, p. 184]:

$$\to H^p_{\mathcal{M}}(X,\mathbb{Z}(q)) \to H^p_{\mathcal{M}}(\widetilde{X},\mathbb{Z}(q)) \oplus H^p_{\mathcal{M}}(\Sigma,\mathbb{Z}(q)) \to H^p_{\mathcal{M}}(E,\mathbb{Z}(q)) \to$$

If $X_\bullet \to X$ is a cubical hyperresolution of X, the motivic cohomology of X can be obtained from the motivic cohomology of $X_\bullet$ using descent:

Lemma 3.3.7 *Let X be a quasi-projective variety. Then, the morphism $H^p_{\mathcal{M}}(X,\mathbb{Z}(q)) \to H^p_{\mathcal{M}}(X_\bullet,\mathbb{Z}(q))$ is an isomorphism.*

Proof. The procedure is via induction in the 2-resolutions of a cubical hyperresolution, as in [Han00, Theo. 2.3]. Let $X_\bullet = \mathrm{red}(X^1_\bullet, X^2_\bullet \ldots, X^r_\bullet) \to X$ be a cubical hyperresolution of X, of lenght r. Then, $X^1_\bullet \to X$ is a 2-resolution given by the following square

$$\begin{array}{ccc} E & \longrightarrow & \widetilde{X} \\ \downarrow & & \downarrow \\ \Sigma & \longrightarrow & X \end{array}$$

The $\square^+_{r-1} \times \square^+_0$-scheme $\mathrm{red}(X^2_\bullet, \ldots, X^r_\bullet)$ is a hyperresolution of the $\square^+_0$-scheme $X^1_{1\bullet} = (E \to \Sigma)$, its consists of $E_\bullet$ and $\Sigma_\bullet$, augmented over E and Σ respectively, this is a diagram of the form

$$\begin{array}{ccccc} E_\bullet & \longrightarrow & E & \longrightarrow & \widetilde{X} \\ \downarrow & & \downarrow & & \downarrow \\ \Sigma_\bullet & \longrightarrow & \Sigma & \longrightarrow & X. \end{array}$$

By cdh-descent, we have the following exact sequence

$$\to H^{p-1}_{\mathcal{M}}(E,\mathbb{Z}(q)) \to H^p_{\mathcal{M}}(X,\mathbb{Z}(q)) \to H^p_{\mathcal{M}}(\widetilde{X},\mathbb{Z}(q)) \oplus H^p_{\mathcal{M}}(\Sigma,\mathbb{Z}(q)) \to$$

By induction hypothesis, the morphisms induced by the augmentation

$$H^{p-1}_{\mathcal{M}}(E,\mathbb{Z}(q)) \to H^{p-1}_{\mathcal{M}}(E_\bullet,\mathbb{Z}(q)) \quad \text{and} \quad H^p_{\mathcal{M}}(\Sigma,\mathbb{Z}(q)) \to H^p_{\mathcal{M}}(\Sigma_\bullet,\mathbb{Z}(q))$$

are isomorphisms. Hence by five-lemma the augmentation $X_\bullet \to X$ induces an isomorphism $H^p_{\mathcal{M}}(X, \mathbb{Z}(q)) \to H^p_{\mathcal{M}}(X_\bullet, \mathbb{Z}(q))$. $\square$

For $X \in \mathbf{Sm}(k)$ and k perfect, if $\mathcal{F}$ is a homotopy invariant sheaf with transfers on the cdh-topology [FV00, Theo. 5.5]:

$$\mathbb{H}^q_{\mathrm{cdh}}(X, \mathcal{F}_{\mathrm{cdh}}) = \mathbb{H}^q_{\mathrm{Nis}}(X, \mathcal{F}_{\mathrm{Nis}}) = \mathbb{H}^q_{\mathrm{Zar}}(X, \mathcal{F}_{\mathrm{Zar}}).$$

Therefore, if X is smooth and k a perfect field, motivic cohomology is computed by Zariski hypercohomology:

$$H^q_{\mathcal{M}}(X, \mathbb{Z}(j)) \cong \mathbb{H}^q_{\mathrm{Zar}}(X, \mathbb{Z}^{SF}(j)_{\mathrm{Zar}}) = \mathrm{Hom}_{\mathrm{DM_{gm}}(k)}\big(M(X), \mathbb{Z}(j)[q]\big).$$

In general if k admits resolution of singularities, for any scheme X of finite type over k, the motivic cohomology has the following presentation:

$$H^q_{\mathcal{M}}(X, \mathbb{Z}(j)) = \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{-}(k)}\big(M(X), \mathbb{Z}(j)[q]\big).$$

3.3.8 The presheaf of Bloch's cycle complex. In the smooth case, the isomorphism (2.2.9) between motivic cohomology and higher Chow groups is given by the relation between Bloch's cycle complexes and Friedlander-Suslin complexes. For this, we need a "sheaffified" version of Bloch's cycle complex as follows:

$$U \mapsto \mathcal{Z}^r(U \times \mathbb{A}^r, \bullet)$$

for $U \in X_{\mathrm{Zar}}$. Then, we have:

Proposition 3.3.9 *For a scheme X of finite type over a field k, we have a quasi-isomorphism of complexes*

$$\mathcal{Z}^r(X \times \mathbb{A}^r, \bullet) \xrightarrow{\mathrm{qis}} R\Gamma(X_{\mathrm{Zar}}, \mathcal{Z}^r(- \times \mathbb{A}^r, \bullet)).$$

Proposition 3.3.10 ([MVW06, Theo. 19.8]) *For $X \in \mathbf{Sm}(k)$, the morphism*

$$\mathbb{Z}^{SF}(r)[2r](X) \to \mathcal{Z}^r(X \times \mathbb{A}^r, \bullet)$$

is a quasi-isomorphism of complexes of Zariski sheaves.

These quasi-isomorphis define the following isomorphism:

$$\begin{aligned}\mathrm{CH}^r(X, m) &\cong \mathrm{CH}^r(X \times \mathbb{A}^r, m) \cong H^{-m}_{\mathrm{Zar}}(X, \mathcal{Z}^r(X \times \mathbb{A}^r, \bullet)) \\ &\cong H^{2r-m}_{\mathrm{Zar}}(X, \mathbb{Z}^{SF}(r)[2r](X)) \cong H^{2r-m}_{\mathcal{M}}(X, \mathbb{Z}(r)).\end{aligned}$$

In the singular case, one way to compute Voevodsky's motivic cohomology is via Hanamura's Chow cohomology groups. In fact, we have the following comparison theorem:

Theorem 3.3.11 *Let X be a quasi-projective variety over a field k. Then, there exists an isomorphism*

$$\mathrm{CHC}^q(X,n) \cong H^{2q-n}_{\mathcal{M}}\big(X,\mathbb{Q}(q)\big).$$

Proof. Let $X_\bullet \to X$ be an augmented cubical hyperresolution with smooth components X_p. By [MVW06, Theo. 19.8], there is a quasi-isomorphism of complexes of Zariski sheaves:

$$\underline{C}_*\mathcal{Z}_{\mathrm{equi}}(\mathbb{A}^r,0)(X_p) = \mathbb{Z}^{SF}(r)[2r](X_p) \xrightarrow{\mathrm{qis}} \mathcal{Z}^r(X_p \times \mathbb{A}^r, \bullet).$$

The compatibility with the face morphisms induces a quasi-isomorphism of total complexes

$$\mathbb{Z}^{SF}_{X_\bullet}(r)[2r] \xrightarrow{\mathrm{qis}} \mathcal{Z}^r(X_\bullet \times \mathbb{A}^r)^*.$$

There are two convergent spectral sequences

$$\begin{array}{rcl} E_1^{p,q}(r) := \mathrm{CH}^r(X_p \times \mathbb{A}^r, -q) & \Rightarrow & \mathrm{CHC}^r(X \times \mathbb{A}^r, p-q) \\ E_1^{p,q}(r) := H^{2r+q}_{\mathcal{M}}\big(X_p, \mathbb{Q}(r)\big) & \Rightarrow & H^{2r-p+q}_{\mathcal{M}}\big(X_\bullet, \mathbb{Q}(r)\big) \end{array}$$

with an isomorphism of total cohomologies

$$\mathrm{CHC}^r(X \times \mathbb{A}^r, p-q) \cong H^{2r-p+q}_{\mathcal{M}}\big(X_\bullet, \mathbb{Q}(r)\big).$$

By homotopy property $\mathrm{CH}^r(X_p \times \mathbb{A}^r, -q) \cong \mathrm{CH}^r(X_p, -q)$, and since $X_\bullet \times \mathbb{A}^r$ defines an hyperresolution of $X \times \mathbb{A}^r$. Via Hanamura's spectral sequence, we have an isomorphism $\mathrm{CHC}^r(X, p-q) \cong \mathrm{CHC}^r(X \times \mathbb{A}^r, p-q)$. □

3.4. Voevodsky's mixed motives

The existence of a category of mixed motives is still conjectural, despite this there are several candidates of triangulated categories of mixed motives. The constructed triangulated tensor category is expected to be the derived category of the conjectural category of mixed motives, for this it is expected that a t-structure can be extracted, so that the heart is the category of mixed motives (with the properties expected by Deligne and Beilinson) [Bei87]. In this direction, there are various constructions:

- Hanamura [HanI, HanII, HanIII]
- Levine [Lev98]
- Voevodsky [Voe00]

which should ideally satisfy the expected properties by the Beilinson's conjectures of the derived category of abelian category of mixed motives. Although the construction of mixed motives is still conjectural, the category does give rise to the groups of motivic cohomology, as can be seen in the previous section, which satisfies all the expected properties [Bei87], [Lic84]. In this section we will give a brief overview of the construction of Voevodsky's motives, the most influential triangulated category of motives. Namely, we will give the construction of geometric motives $\mathrm{DM}_{\mathrm{gm}}(k)$, as well as a sheaf-theoric construction of the motivic complexes $\mathrm{DM}_{-}^{\mathrm{eff}}(k)$. In [Han00] and [Lev98], Hanamura and Levine respectively apply the technique of cubical hyperresolutions in the context of mixed motives to extend the definition of motives to schemes of finite type over k. They give two other approximations of the triangulated category of mixed motives over a field that admits resolution of singularities.

3.4.1 Voevodsky's mixed motives. Let k be a perfect field that admits resolution of singularities, and $\mathbf{Sm}(k)$ the category of smooth schemes of finite type over k. For this construction, see [MVW06] and [VSF00].

Definition 3.4.2 Let $X, Y \in \mathbf{Sm}(k)$. The group $\mathrm{Corr}_{\mathrm{fin}}(X, Y)$ of **finite correspondences** form X to Y is the abelian group generated by integral subschemes $Z \subset X \times Y$ such that:

- the projection $\mathrm{pr}_X \colon Z \to X$ is finite
- the image $\mathrm{pr}_X(Z) \subset X$ is an irreducible component of X.

The composition of correspondences is given as follows: For $X, Y, W \in \mathbf{Sm}(k)$, and $\alpha \in \mathrm{Corr}_{\mathrm{fin}}(X, Y)$, $\beta \in \mathrm{Corr}_{\mathrm{fin}}(Y, W)$. Suppose that X and Y are irreducible. Then each irreducible component C of $|\alpha| \times W \cap X \times |\beta|$ is finite over X and $\mathrm{pr}_X(C) = X$. Then, the composition is

$$\alpha \circ \beta := \mathrm{pr}_{*XW}\big(\mathrm{pr}_{XY}^{*}(\alpha) \cdot \mathrm{pr}_{YW}^{*}(\beta)\big).$$

The category of $\mathbf{Corr}_{\mathrm{fin}}(k)$ is the category with the same objects as $\mathbf{Sm}(k)$ and morphisms

$$\mathbf{Corr}_{\mathrm{fin}}(k)(X, Y) := \mathrm{Corr}_{\mathrm{fin}}(X, Y)$$

with the composition defined above. There exists a functor

$$\mathbf{Sm}(k) \to \mathbf{Corr}_{\mathrm{fin}}(k)$$

that sends X to $[X]$ (viewed as an object in $\mathbf{Corr}_{\mathrm{fin}}(k)$) and a morphism $f\colon X \to Y$ to $\Gamma_f = f_* \in \mathrm{Corr}_{\mathrm{fin}}(X,Y)$. The product $\times_k$ on $\mathbf{Sm}(k)$ induces a product on $\mathbf{Corr}_{\mathrm{fin}}(k)$. This makes $\mathbf{Corr}_{\mathrm{fin}}(k)$ an additive and tensor category. Consider the bounded homotopy category $\mathbf{K}^b\big(\mathbf{Corr}_{\mathrm{fin}}(k)\big)$, this a tensor triangulated category.

Definition 3.4.3 The category $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$ of **effective geometric motives** is obtained in the following way:

(1) Localize $\mathbf{K}^b\big(\mathbf{Corr}_{\mathrm{fin}}(k)\big)$ with respect to the thick subcategory generated by complexes of the form:

  (a) **Homotopy.** $\mathrm{pr}_{X*}\colon [X \times \mathbb{A}^1] \to [X]$

  (b) **Mayer-Vietoris.** $[U \cap V] \to [U] \oplus [V] \to [X]$, where U and V are Zariski open subschemes of X such that $X = U \cup V$.

(2) The category $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$ is the pseudo-abelian completion of the resulting quotient category.

The category $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$ then is triangulated tensor category [BS01]. Finally, the category of geometric motives $\mathrm{DM}_{\mathrm{gm}}(k)$ is obtained by inverting the *Lefschetz motive*. For any $X \in \mathbf{Sm}(k)$, the structural morphism $X \to \mathrm{Spec}(k)$ gives us in $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$ a morphism $M_{\mathrm{gm}}(X) \to \mathbb{Z}$. There is a canonical dintinguished triangle

$$\widetilde{M}_{\mathrm{gm}}(X) \to M_{\mathrm{gm}}(X) \to \mathbb{Z} \to \widetilde{M}_{\mathrm{gm}}(X)[+1]$$

where $\widetilde{M}_{\mathrm{gm}}(X)$ is the reduced motive of X determined in $\mathbf{K}^b\big(\mathbf{Corr}_{\mathrm{fin}}(k)\big)$ by the complex $[X] \to \mathrm{Spec}(k)$. For a k-rational point in X, there is a decomposition $M_{\mathrm{gm}}(X) = \mathbb{Z} \oplus \widetilde{M}_{\mathrm{gm}}(X)$ in $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$. The **Tate motive** is $\mathbb{Z}(1) := \widetilde{M}_{\mathrm{gm}}(\mathbb{P}^1)[2]$. For $n \geq 0$, set $\mathbb{Z}(n) := \mathbb{Z}(1)^{\otimes n}$. For any object $M \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$, the **Tate twist** is given by $M(n) := M \otimes \mathbb{Z}(n)$.

Definition 3.4.4 The category $\mathrm{DM}_{\mathrm{gm}}(k)$ is obtained from $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$ by inverting $\mathbb{Z}(1)$. More precisely, objects of $\mathrm{DM}_{\mathrm{gm}}(k)$ are pairs of the form (A,p) where $A \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$ and $p \in \mathbb{Z}$. The morphisms are given by

$$\mathrm{DM}_{\mathrm{gm}}(k)\big((A,p),(B,q)\big) := \lim_{r \geq -p,-q} \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)\big(A(r+p), B(r+q)\big).$$

The resulting category $\mathrm{DM}_{\mathrm{gm}}(k)$ is a rigid tensor linear triangulated category [MVW06, Theo. 20.17]. The four steps in the construction of $\mathrm{DM}_{\mathrm{gm}}(k)$ can be summarized in the following diagram:

$$\mathbf{Sm}(k) \to \mathbf{Corr}_{\mathrm{fin}}(k) \to \mathbf{K}^b\big(\mathbf{Corr}_{\mathrm{fin}}(k)\big) \to \mathbf{K}^b\big(\mathbf{Corr}_{\mathrm{fin}}(k)\big)/\langle HI + MV\rangle$$

$$\xrightarrow{\text{pse. ab}} \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k) \xrightarrow{\mathbb{Z}(1)^{-1}} \mathrm{DM}_{\mathrm{gm}}(k).$$

3.4.5 Properties of $\mathrm{DM}_{\mathrm{gm}}(k)$**.** The main properties of the category $\mathrm{DM}_{\mathrm{gm}}(k)$ are:

- **Kunneth formula:** $M(X \times Y) = M(X) \otimes M(Y)$.
- **Homotopy invariance:** $M(X \times \mathbb{A}^1) = M(X)$.
- **Mayer-Vietoris:** For $X = U \cup V$, there is a distinguished triangle
$$M(U \cap V) \to M(U) \oplus M(V) \to M(X) \to M(U \cap V)[1].$$
- **Projective bundle:** Let $\mathcal{E}$ be a vector bundle over X of rank $n+1$. Consider $E = \mathbb{P}(\mathcal{E}) \to X$ its projectivization, then we have:
$$M(E) \cong \bigoplus_{i=0}^{n} M(X) \otimes \mathbb{Z}(i)[2i].$$
- **Blow-up triangle:** Let $Z \subset X$ be a smooth closed subscheme, $\mathrm{Bl}_Z X$ the blow-up of X along Z with exceptional divisor E. Then, there is a distinguished triangle
$$M(E) \to M(\mathrm{Bl}_Z X) \oplus M(E) \to M(X) \to M(E)[1].$$
- **Gysin triangle:** If $Z \subset X$ is a smooth closed subscheme of codimension c in X, there exist a distinguished triangle
$$M(X - Z) \to M(X) \to M(Z)(c)[2c] \to M(X - Z)[1].$$
- **Duality:** The category $\mathrm{DM}_{\mathrm{gm}}(k)$ is rigid, i.e. there is a duality functor $(-)^{\vee} \colon \mathrm{DM}_{\mathrm{gm}}(k) \to \mathrm{DM}_{\mathrm{gm}}(k)^{\mathrm{opp}}$. For X smooth and projective of dimension d, we have
$$M(X)^{\vee} = M(X)(-d)[-2d].$$

– **Adjunction:** The functor $- \otimes B^\vee$ is right adjoint of $- \otimes B$.

Remark 3.4.6 The construction of $\mathrm{DM}_{\mathrm{gm}}(k)$ is based on the model of the construction of $\mathrm{Mot}_{\mathrm{rat}}(k)$, the Grothendieck pure motives [MNP13]. The construction of $\mathrm{DM}_{\mathrm{gm}}(k)$ being covariant, contrary to $\mathrm{Mot}_{\mathrm{rat}}(k)$, which was cohomological. The relation between these categories is the following: there exists a fully faithful embedding

$$\mathrm{Mot}_{\mathrm{rat}}(k) \hookrightarrow \mathrm{DM}_{\mathrm{gm}}(k)$$

this is given by the fact that

$$\mathrm{DM}_{\mathrm{gm}}(k)(M(X), M(Y)) \cong \mathrm{CH}^{d_Y}(X \times Y) = \mathrm{Corr}^0_{\mathrm{rat}}(Y, X).$$

Motivic Complexes

The deepest properties, such as Blow-up, Gysin and Duality, are proved via an embedding of $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$ into a triangulated category of sheaves with transfers $\mathrm{DM}^{\mathrm{eff}}_{-}(k)$, the category of *motivic complexes*. We will give here a sketch of Voevodsky's construction.

Definition 3.4.7 A **presheaf with transfers** is a contravariant additive functor $\mathcal{F} \colon \mathbf{Corr}_{\mathrm{fin}}(k) \to \mathbf{Ab}$. The presheaf is called **homotopy invariant** if the natural morphism $\mathrm{pr}_X^* \colon \mathcal{F}(X) \to \mathcal{F}(X \times \mathbb{A}^1)$ is an isomorphism.

Example 3.4.8 The prototype of a presheaf with transfers is $\mathrm{Corr}_{\mathrm{fin}}(-, X)$.

The category of presheaf with transfers $\mathbf{PST}(k)$ forms an abelian category with enough injectives/projectives. Nisnevich (Zariski, étale) covering generate a Grothendieck topology, the Nisnevich (Zariski, étale) topology on $\mathbf{Sch}(k)$.

Definition 3.4.9 A presheaf with transfers $\mathcal{F}$ is called a **Nisnevich sheaf with transfers** (Zariski, étale) if the composition $\mathbf{Sm}(k) \to \mathbf{Corr}_{\mathrm{fin}}(k) \to \mathbf{Ab}$ is a sheaf for the Nisnevich (Zariski, étale) topology.

Denote the category of Nisnevich sheaf with transfers by $\mathbf{Nis}_{\mathrm{tr}}(k)$. Again, the category $\mathbf{Nis}_{\mathrm{tr}}(k)$ is an abelian category with enough injectives. There are fully faithful inclusions $\mathbf{\acute{e}t}_{\mathrm{tr}}(k) \subseteq \mathbf{Nis}_{\mathrm{tr}}(k) \subseteq \mathbf{Zar}_{\mathrm{tr}}(k) \subseteq \mathbf{PST}(k)$.

Definition 3.4.10 Let $\mathbf{D}^-(\mathbf{Nis}_{\mathrm{tr}}(k))$ be the derived category of bounded above complexes of Nisnevich sheaves with transfers. The category $\mathrm{DM}^{\mathrm{eff}}_-(k)$ of **effective motivic complexes** is the full subcategory of $\mathbf{D}^-(\mathbf{Nis}_{\mathrm{tr}}(k))$ of bounded above complexes of Nisnevich sheaf with transfers whose cohomology sheaves are homotopy invariant.

3.4.11 Let $\mathcal{F}$ be a presheaf with transfers. The **Suslin complex** $C_*(\mathcal{F})$ is the complex of presheaves on $\mathbf{Sm}(k)$ defined by

$$C_n(\mathcal{F})(U) := \mathcal{F}(U \times \Delta^n)$$

where the differentials are alternanting sums of pullbacks to the faces. By definition the sheaf $C_n(\mathcal{F})$ is placed in degree $-n$, making $C_*(\mathcal{F})$ a complex that is bounded above, let $C^*(\mathcal{F})$ the complex defined by the change of indices $C^n(\mathcal{F}) := C_{-n}(\mathcal{F})$. If $\mathcal{F}$ is a Nisnevich sheaf with transfers, the Suslin complex $C_*(\mathcal{F})$ has homotopy invariant cohomology sheaves [And04, Cor. 19.2.5.2]. This construction defines a functor $C_*\colon \mathbf{Nis}_{\mathrm{tr}}(k) \to \mathrm{DM}^{\mathrm{eff}}_-(k)$. Now we give a way to build sheaves with transfers, this allows us to associate a motivic complex to any variety X as follows. We denote by $\mathbb{Z}_{\mathrm{tr}}(X)$ the presheaf with transfers defined by

$$\mathbb{Z}_{\mathrm{tr}}(X)(U) := \mathrm{Corr}_{\mathrm{fin}}(U, X)$$

with $U \in \mathbf{Sm}(k)$, this defines a functor $\mathbb{Z}_{\mathrm{tr}}(-)\colon \mathbf{Sch}(k) \to \mathbf{PST}(k)$. We can see that the presheaf $\mathbb{Z}_{\mathrm{tr}}(X)$ is an étale sheaf with transfers [MVW06, Lemma 6.2]. Hences $\mathbb{Z}_{\mathrm{tr}}$ induces a functor

$$\mathbb{Z}_{\mathrm{tr}}\colon \mathbf{K}^b\big(\mathbf{Corr}_{\mathrm{fin}}(k)\big) \to \mathbf{D}^-(\mathbf{Nis}_{\mathrm{tr}}(k)).$$

The **motivic complex** associated to X is the class $C_*(X) := C_*(\mathbb{Z}_{\mathrm{tr}}(X))$ in $\mathrm{DM}^{\mathrm{eff}}_-(k)$. Explicitly, $C_n(X)$ is the presheaf in $\mathbf{Corr}_{\mathrm{fin}}(k)$ given by

$$C_n(X)(U) = \mathrm{Corr}_{\mathrm{fin}}(U \times \Delta^n, X).$$

In particular, if X is smooth, then $\mathbb{Z}_{\mathrm{tr}}(X)$ is the presheaf in $\mathbf{Corr}_{\mathrm{fin}}$ represented by X.

Theorem 3.4.12 (**Localization Theorem**, [Voe00, 3.2.3]) *The functor C_* extends to a functor*

$$RC_*\colon \mathbf{D}^-(\mathbf{Nis}_{\mathrm{tr}}(k)) \to \mathrm{DM}^{\mathrm{eff}}_-(k)$$

which is left adjoint to the natural embedding. The functor RC_* identifies $\mathrm{DM}^{\mathrm{eff}}_-(k)$ with the localization of $\mathbf{D}^-(\mathbf{Nis}_{\mathrm{tr}}(k))$ with respect to the thick subcategory generated by complexes of the form

$$\mathbb{Z}_{\mathrm{tr}}(X \times \mathbb{A}^1) \to \mathbb{Z}_{\mathrm{tr}}(X), \quad \text{for} \quad X \in \mathbf{Sm}(k).$$

Theorem 3.4.13 (**Embedding Theorem**, [Voe00, 3.2.6]) *The composition functor*

$$RC_* \circ \mathbb{Z}_{\mathrm{tr}} \colon \mathbf{K}^b\big(\mathbf{Corr}_{\mathrm{fin}}(k)\big) \to \mathrm{DM}_-^{\mathrm{eff}}(k)$$

factors through $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k)$*, and the functor* $i\colon \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k) \to \mathrm{DM}_-^{\mathrm{eff}}(k)$ *is a fully faithful embedding. In fact, we have a commutative diagram*

$$\begin{tikzcd} \mathbf{K}^b\big(\mathbf{Corr}_{\mathrm{fin}}(k)\big) \arrow[r, "\mathbb{Z}_{\mathrm{tr}}"] \arrow[d, "M"'] & D^-(\mathbf{Nis}_{\mathrm{tr}}(k)) \arrow[d, "RC_*"] \\ \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k) \arrow[r, "i"] & \mathrm{DM}_-^{\mathrm{eff}}(k) \end{tikzcd}$$

where the Suslin complex $C_*(X) = RC_*(\mathbb{Z}_{\mathrm{tr}}(X)) = i(M(X))$*.*

3.4.14 The motives $M(X)$ and $M^c(X)$ [Voe00, Sect. 4]. For $X \in \mathbf{Sch}(k)$, we define the motive $M(X)$ as the class of $C_*(X)$ in $\mathrm{DM}_-^{\mathrm{gm}}(k)$. By embedding theorem, this construction extends the definition of $M(X)$ for $X \in \mathbf{Sm}(k)$ given in (3.4.3). For motives with compact support, consider the following definition. Let $X \in \mathbf{Sm}(k)$ and $Y \in \mathbf{Var}(k)$, the group $\mathrm{Corr}_{\mathrm{q-fin}}(X,Y)$ of **quasi-finite correspondences** from X to Y is the abelian group generated by integral subschemes $Z \subset X \times Y$ such that $\mathrm{pr}_X \colon Z \to X$ is quasi-finite over an irreducible component of X. This allows us to consider the following Nisnevich sheaf with transfers, for $X \in \mathbf{Sch}(k)$:

$$\mathbb{Z}_{\mathrm{tr}}^c(X)(U) := \mathrm{Corr}_{\mathrm{q-fin}}(U,X).$$

Then, the **motivic complex with compact support** of X is defined by $C_*^c(X) := C_*(\mathbb{Z}_{\mathrm{tr}}^c(X))$. The **motive with compact support** $M^c(X)$ is given as the class of $C_*^c(X)$ in $\mathrm{DM}_-^{\mathrm{gm}}(k)$.

Using the blow-up and Gysin distinguished triangles we have the following important result:

Corollary 3.4.15 ([Voe00, Cor. 4.1.4, 4.1.6]) *Let k a field which admits resolution of singularities. Then for any scheme X of finite type over k, the motives $M(X)$ and $M^c(X)$ belong to the triangulated category* $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k)$*.*

3.4.16 Comparison theorems. We denote by $\mathcal{DM}(k)$ the triangulated category of motives constructed in [Lev94]. In $\mathrm{char}(k) = 0$, Levine [Lev94, Part 1, Chap. VI, Theo. 2.5.5] proves that there is an equivalence of triangulated categories $\mathcal{DM}(k) \to \mathrm{DM}_{\mathrm{gm}}(k)$ between Levine's and Voevosky's motives. The comparison between Hanamura's [HanI] and Voevodsky's motives is due to M. Bondarko [Bon09]. Bondarko proves that there exists an (anti-)equivalence of triangulated categories of motives $\mathcal{D}(k) \to \mathrm{DM}_{\mathrm{gm}}(k)$.

Chapter 4

A regulator morphism for singular varieties

The KLM-formula is first defined for projective and smooth varieties at complex level, in terms of cubical Bloch's cycle complex and currents, which defines the motivic cohomology and absolute Hodge comology respectively [KLM06]. In [KL07], the authors extend this formula for open smooth varieties, and complete normal crossing varieties. This formula admits extension to complete singular varieties, then all quasi-projective varieties. In this generalization, Higher Chow groups is a Borel-Moore homology theory and is replaced by Voevodsky's motivic cohomology, which is identified with higher Chow cohomology groups. The procedure is via cubical hyperresolutions [GNPP88] and [GNA02]. In this chapter we will describe explicitly the complexes that define the regulator morphism in the singular case:

$$\mathrm{reg}_X \colon \mathrm{CHC}^r(X,m) \cong H^{2r-m}_{\mathcal{M}}(X,\mathbb{Q}(r)) \to H^{2r-m}_{\mathcal{H}}(X,\mathbb{Q}(r)).$$

This morphism is first described in terms of double (triple) complexes. Then the cycle-class morphism is given by the cohomology of the total complexes. The Abel-Jacobi morphism is evaluated on the graded pieces through the spectral sequence, and described page to page by the morphism of spectral sequences between motivic cohomology and absolute Hodge cohomology.

4.1. The regulator for a complete NCD

Consider the case where $Y \subset X$ is a complete normal crossing divisor. For $Y = \bigcup_i^N Y_i$, we consider the following notation:

- $Y_I = \bigcap_{i\in I} Y_i$, $I = \{i_1,\dots,i_\ell\}$,
- $Y_{[\ell]} = \coprod_{|I|=\ell} Y_I$,
- $j_{I,j} \colon Y_{I\cap\{j\}} \hookrightarrow Y_I$,
- $Y^I = \bigcup_{j\notin I} Y_{I\cap\{j\}} \subset Y_I$.

This produces a semi-simplicial hyperresolution $a\colon Y_\bullet \to Y$:

$$\cdots \overrightarrow{\underrightarrow{\rightrightarrows}} Y_{[3]} \overrightarrow{\rightrightarrows} Y_{[2]} \rightrightarrows Y_{[1]} \xrightarrow{a} Y$$

with smooth and projective components, and a satisfies the cohomological descent. To construct the cycle class morphism associated to Y, we need to introduce double complexes to give us spectral sequences that compute motivic cohomology and Deligne cohomology. According to [KLM06, KL07] and [GGK10], there is a fourth quadrant double complex:

$$\mathcal{Z}_Y^{i,j}(r) := \mathcal{Z}^r(Y_{[i]}, -j) := \bigoplus_{|I|=i} \mathcal{Z}_{\mathbb{R}}^r(Y_I, -j)_{Y^I}$$

with differentials $\partial_B\colon \mathcal{Z}_Y^{i,j}(r) \to \mathcal{Z}_Y^{i,j+1}(r)$ (vertical, Bloch) and horizontal $\partial_J\colon \mathcal{Z}_Y^{i,j}(r) \to \mathcal{Z}_Y^{i+1,j}(r)$. The associated total complex is

$$\mathcal{Z}_Y^\bullet(r) := \mathbf{s}^\bullet \mathcal{Z}_Y^{\bullet,\bullet}(r) \qquad \text{and} \qquad \partial = \partial_B \pm \partial_J.$$

The motivic cohomology of Y is given by

$$H_{\mathcal{M}}^{2r+*}(Y, \mathbb{Z}(r)) := H^*(\mathcal{Z}_Y^\bullet(r), \partial).$$

For Deligne cohomology, we consider the following double complex

$$K_Y^{i,j}(r) := B_Y^{i,j}(r) \oplus F^r D_Y^{i,j}(r) \oplus D_Y^{i,j-1}(r) := \bigoplus_{|I|=i} C_{\mathcal{D}}^{2r+j}(Y_I, \mathbb{Q}(r))_{Y^I}$$

$$:= \bigoplus_{|I|=i} \left\{ C_{\#}^{2r+j}(Y_I, \mathbb{Q}(r)) \oplus F^r \mathcal{D}_{\#}^{2r+j}(Y_I) \oplus \mathcal{D}_{\#}^{2r+j-1}(Y_I) \right\}$$

where $C_{\#}^\bullet(Y_I, \mathbb{Q}(r)) := \mathcal{I}^\bullet\{Y^I\}(Y_I) \otimes_{\mathbb{Z}} \mathbb{Q}(r)$ are the locally intersection currents [KL07, Def. 8.5], and $\mathcal{D}_{\#}^\bullet(Y_I) := \mathcal{N}\{Y^I\}(Y_I)$ denotes the (locally) normal currents. The associated total complex is

$$C_{\mathcal{D},Y}^\bullet(r) := \mathbf{s}^\bullet K_Y^{\bullet,\bullet}(r).$$

The Deligne cohomology is defined by

$$H_{\mathcal{D}}^{2r+*}(Y, \mathbb{Q}(r)) := H^*(Y, \mathbf{s}^\bullet K_Y^{\bullet,\bullet}(r)) = H^*(Y_\bullet, K_Y^{\bullet,\bullet}(r)).$$

The KLM-currents produces a morphism of double complexes

$$\begin{array}{rcl} \mathcal{Z}_Y^{i,j}(r) & \to & K_Y^{i,j}(r) \\ \alpha & \mapsto & (-2\pi i)^{r+j}\big((2\pi i)^{-j} T_\alpha, \Omega_\alpha, R_\alpha\big). \end{array}$$

This is a moprhism on 0^{th}-page of both spectral sequences. Then, the morphism of total complexes:

$$\mathrm{reg}\colon \mathcal{Z}^{\bullet}_{Y}(r) \to C^{\bullet}_{\mathcal{D},Y}(r)$$

is the regulator morphism for Y on the level of complexes. The cycle-class morphism of total cohomologies

$$\mathrm{cl}_Y\colon H^{2r+*}_{\mathcal{M}}(Y,\mathbb{Q}(r)) \to H^{2r+*}_{\mathcal{D}}(Y,\mathbb{Q}(r))$$

is compatible with natural weight filtration arising from the double complexes.

4.2. The absolute Hodge cohomology

Sheaves, cohomology and cohomological descent

Let X be a quasi-projective variety over $\mathbb{C}$. Consider a cubical hyperresolution $\mathrm{red}(X^1_\bullet,\ldots,X^r_\bullet) \to X$, with associated augmented semi-simplicial scheme $X_\bullet \to X$ [GNPP88]. *Cohomological descent* is a technique introduced by Deligne [HodgeIII], to extend the notion of mixed Hodge structures to non-smooth varieties. This is a variant of the technique given by Čech. For example, this property allows us for a 2-resolution

$$\begin{array}{ccc} E & \xrightarrow{j} & \widetilde{X} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ X_{\mathrm{sing}} & \xrightarrow{i} & X \end{array}$$

to obtain a blow-up exact sequence on cohomology. For a sheaf $\mathcal{F}$ on X, we have an exact sequence

$$\cdots \to H^i(X,\mathcal{F}) \to H^i(\widetilde{X},p^*\mathcal{F}) \oplus H^i(X_{\mathrm{sing}},i^*\mathcal{F}) \to H^i(E,(p\circ j)^*\mathcal{F}) \to \cdots$$

In general, a sheaf $\mathcal{F}^\bullet$ on a semi-simplicial scheme $X_\bullet$ is a collection of sheaves $\mathcal{F}^p$ on X_p, and morphisms of sheaves $d_i\colon d_i^*\mathcal{F}^{q-1} \to \mathcal{F}^q$ satisfying the semi-simplicial identities. A morphism of sheaves $F\colon \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ on $X_\bullet$ is a family of morphisms $\{F^p\colon \mathcal{F}^p \to \mathcal{G}^p\}$ compatible with the face morphisms. For example, for a sheaf $\mathcal{F}^\bullet$ on $X_\bullet$, the Godement resolutions $\mathcal{C}^\bullet_{\mathrm{Gdm}}(\mathcal{F}^p)$ give injective resolutions of $\mathcal{F}^p$, and fit together to give an injective resolution $\mathcal{C}^\bullet_{\mathrm{Gdm}}(\mathcal{F}^\bullet)$ of $\mathcal{F}^\bullet$ on $X_\bullet$. This allows us to define the cohomology of a semi-simplicial scheme $X_\bullet$ with values on $\mathcal{F}^\bullet$. The abelian groups

$$F^{q,p} := \Gamma(X_p, \mathcal{C}^q_{\mathrm{Gdm}}(\mathcal{F}^p))$$

form part of a double complex. We define

$$H^k(X_\bullet, \mathcal{F}^\bullet) := H^k(s^\bullet(F^{\bullet,\bullet})).$$

For an augmentation $a\colon X_\bullet \to X$ and a sheaf $\mathcal{F}^\bullet$ on $X_\bullet$, the sheaves $a_*\mathcal{C}^q_{\mathrm{Gdm}}(\mathcal{F}^p)$ form a double complex of sheaves on X; its associated total complex defines

$$Ra_*\mathcal{F}^\bullet := \mathbf{s}^\bullet\big[a_*\mathcal{C}^\bullet_{\mathrm{Gdm}}(\mathcal{F}^\bullet)\big].$$

This complex computes the cohomology of $X_\bullet$, and we have

$$\mathbb{H}^k(X, Ra_*\mathcal{F}^\bullet) = H^k(X_\bullet, \mathcal{F}^\bullet).$$

In general, we have natural adjunction morphisms of sheaves on X

$$\mathcal{F} \to Ra_*(a^*\mathcal{F}).$$

Definition 4.2.1 An augmented semi-simplicial scheme $a\colon X_\bullet \to X$ has the property of **cohomological descent** if the natural morphism

$$\mathcal{F} \to Ra_*(a^*\mathcal{F})$$

is an isomorphism in $\mathbf{D}^+(\mathbb{Q}_X)$, for any sheaf $\mathcal{F}$ on X.

In this case, we have a spectral sequence

$$E_1^{p,q} = H^q(X_p, a_p^*\mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F}).$$

Remark 4.2.2 For an augmented semi-simplicial hyperresolution $a\colon X_\bullet \to X$, cohomological descent allows us to study the cohomology of a singular variety in terms of the cohomology groups of smooth varieties lying over it.

Mixed Hodge complexes

Mixed Hodge complexes constitute a technique introduced by P. Deligne in [HodgeIII] to extend his theory of Hodge structures to the case of the cohomology of singular varieties, using simplicial resolutions. Recall that a mixed Hodge complex $K^\bullet$ is given by

- a filtered complex $(K^\bullet_{\mathbb{Q}}, W_\bullet)$,
- a bifiltered complex $(K^\bullet_{\mathbb{C}}, W_\bullet, F^\bullet)$,

– a filtered quasi-isomorphism $(K^\bullet_{\mathbb{Q}}, W_\bullet) \otimes \mathbb{C} \to (K^\bullet_{\mathbb{C}}, W_\bullet)$.

satisfying the conditions given in (2.3.1). An important result about mixed Hodge complex is the following theorem [HodgeIII, 8.1.9]:

Theorem 4.2.3 *Let $K^\bullet$ be a mixed Hodge complex. Then $H^*(K^\bullet)$ has a mixed Hodge structure:*

– *The Hodge filtration is induced by*

$$F^p H^m(K^\bullet_{\mathbb{C}}) = \mathrm{Im}\big(H^m(F^p K^\bullet_{\mathbb{C}}) \to H^m(K^\bullet_{\mathbb{C}})\big)$$

moreover, the espectral sequence $(K^\bullet_{\mathbb{C}}, F)$ degenerates at E_1.

– *The weight filtration is defined by*

$$W_{\ell+m}(K^\bullet_{\mathbb{Q}}) = \mathrm{Im}\big(H^m(W_\ell K^\bullet_{\mathbb{Q}}) \to H^m(K^\bullet_{\mathbb{Q}})\big)$$

and the espectral sequences $(K^\bullet_{\mathbb{Q}}, W)$ and (Gr^p_F, W) degenerates at E_2.

Let U be a smooth quasi-projective variety over $\mathbb{C}$. By Nagata [Nag62] and Hironaka's resolution of singularities [Hir64] we can find a compatification $U \hookrightarrow X$ with X smooth projective, and $Y := X - U$ a normal crossing divisor. In [KL07], Kerr and Lewis construct a mixed Hodge complex associate to the pair (X, U), this is a Gysin complex given in terms of the total complexes corresponding to semi-simplicial scheme $Y^{[\bullet]}$, augmented to X.

Theorem 4.2.4 *The following system given by:*

– $(K^\bullet_{\mathbb{Q}}, W) := (\mathbf{s}^\bullet C(r), W_\bullet)$,

– $(K^\bullet_{\mathbb{C}}, W, F) := (\mathbf{s}^\bullet \mathcal{D}(r), W_\bullet, F^\bullet)$

defines a mixed Hodge complex, with filtrations $W_\bullet (= {}'W^{-\bullet})$ and $F^\bullet$ as in (2.5.3). In particular, the complex $R\Gamma(K^\bullet)$ gives us the absolute Hodge cohomology of U.

4.2.5 Guillén-Navarro's extesion criterion [GNA02]. The above construction defines a functor

$$K^\bullet_{\mathcal{H}} \colon \mathbf{Sm}(\mathbb{C}) \to \mathbf{MHC}, \tag{4.1}$$

where the complex

$$R\Gamma_{\mathcal{H}}(K^\bullet) = \mathrm{Cone}\big(\widehat{W}_0 \mathbf{s}^\bullet C(r) \oplus F^0 \widehat{W}_0 \mathbf{s}^\bullet \mathcal{D}(r) \to \widehat{W}_0 \mathbf{s}^\bullet \mathcal{D}(r)\big)[-1]$$

defines the absolute Hodge cohomology of U, and $\widehat{W}$ is Deligne's décalage filtration. Deligne shows that for any complex variety X, we can construct a functorial mixed Hodge complex associated to X:

$$K^{\bullet}_{\mathcal{H}}\colon \mathbf{QuProj}(\mathbb{C}) \to \mathbf{MHC}$$

For this, Deligne in [HodgeIII] extends the notion of mixed Hodge complex to quasi-projective complex algebraic varieties with singularities, using its simplicial hypercoverings. Here we use the alternative technique of cubical hyperresolutions developed in [GNPP88]. In [GNA02], Guillen and Navarro-Aznar developed a general descent theory, aided by the theory of cubical hyperresolutions. This allows to establish an extension criterion of functors defined on the category of smooth varieties, to the category of all varieties and define the weight filtration for any complex variety. This technique permits to extend the above construction, defined in (2.5.1-2.5.3), via the following theorem:

Theorem 4.2.5 ([GNA02, Theorem 2.1.5]) *There is an essentially unique functor*

$$K^{\bullet\prime}_{\mathcal{H}}\colon \mathbf{QuProj}(k) \to \mathbf{K}^b(\mathbf{MHC})$$

extending the functor $K^{\bullet}_{\mathcal{H}}\colon \mathbf{Sm}(\mathbb{C}) \to \mathbf{MHC}$ of (4.1), such that:

(D) *If $X_\bullet$ is an elementary acyclic square (abstract blow-up), then $\mathbf{s}K^{\bullet\prime}_{\mathcal{H}}(X_\bullet)$ is acyclic (abstract blow-up).*

Remark 4.2.6 The extension criterion is the same technique that Hanamura applies in [Han00] to the higher Chow groups. Other applications are in Grothendieck's theory of motives, and the weight filtration in algebraic K-theory. It is also closely related to the conditions of descent in the cdh-cohomology (or h-topology).

4.2.7 Construction. Let U be an arbitrary quasi-pojective variety over $\mathbb{C}$, with compatification $U \hookrightarrow X$. Then, we may take a hyperresolution of the pair $(X_\bullet, U_\bullet) \to (X, U)$, these are hyperresolutions $X_\bullet \to X$ and $U_\bullet \to U$, such that $Y_\bullet := X_\bullet - U_\bullet$ is a simplicial NCD, i.e. $Y_p := X_p - U_p$ is a normal crossing divisor for all p

$$\begin{array}{ccccc} Y_\bullet & \longrightarrow & X_\bullet & \longleftarrow & U_\bullet \\ \downarrow & & \downarrow & & \downarrow \\ Y & \longrightarrow & X & \longleftarrow & U \end{array}$$

The semi-simplicial scheme given by the cubical hyperresolutions $U_\bullet = X_\bullet - Y_\bullet \to U$ satifies the descent cohomological property [GNPP88].

Definition 4.2.8 An $\mathbb{A}$-mixed Hodge complex $K^\bullet$ over $U_\bullet$ consists of:

(i) a filtered complex $(K^\bullet_{\mathbb{Q}}, W_\bullet) \in \mathbf{D}^+\mathbf{F}(U_\bullet, \mathbb{Q})$,

(ii) a bifiltered complex $(K^\bullet_{\mathbb{C}}, W_\bullet, F^\bullet) \in \mathbf{D}^+\mathbf{F}_2(U_\bullet, \mathbb{C})$,

(iii) a filtered quasi-isomorphism $(K^\bullet_{\mathbb{Q}}, W_\bullet) \otimes \mathbb{C} \to (K^\bullet_{\mathbb{C}}, W_\bullet)$ in $\mathbf{D}^+\mathbf{F}(U_\bullet, \mathbb{C})$

such that the restriction of $K_\bullet$ to each U_p is an $\mathbb{A}$-mixed Hodge complex.

In order to construct the absolute Hodge cohomology of U, singular and non-compact, we need to describe explicitly a mixed Hodge complex associated to U. This construction is given component by component in the hyperresolution of U. Applying this construction to each $U_p = X_p - Y_p$, this family of filtered complexes forms a filtered complex over $U_\bullet$, and we obtain a simplicial mixed Hodge complex $K^\bullet \in \mathbf{MHC}(U_\bullet)$. Considering the total (simple) complex, we have a mixed Hodge complex over U.

Proposition 4.2.9 *Let $U_\bullet \hookrightarrow X_\bullet$ be a good compactification of a smooth simplicial variety $U_\bullet$, with $Y_p = X_p - U_p$ a normal crosing divisor. Taking $K^\bullet_{\mathbb{Q}}$ as the simple complex given on each component $X_p - Y_p$ by $K^p_{\mathbb{Q}} = \mathbf{s}^\bullet\mathcal{C}(r)$ as in (2.5.1). In the same way, we define $K^p_{\mathbb{C}} = \mathbf{s}^\bullet\mathcal{D}(r)$. These complexes are given by the resolution $Y_p^{[\bullet]} \to Y_p^{[0]} = X_p$, together with the usual filtrations $W_\bullet$ and $F^\bullet$ forms a simplicial mixed Hodge complex.*

Absolute Hodge cohomology for singular varieties

In characteristic zero, using resolution of singularities we can produce a cdh-covering $X_\bullet \to X$, for any variety X. Then, a natural way to extend a functor (cohomology theory) defined on smoooth schemes to singular varieties is to use cdh-covers. More concretely, hyperresolutions is a way to generate this type of covers. Consider an arbitrary $U \in \mathbf{QuProj}(k)$. The (polarizable) mixed Hodge complex is given in terms of hyperresolution $(X_\bullet, U_\bullet) \to (X, U)$. By definiton, the absolute Hodge cohomology of one component U_p of $U_\bullet$ is computed by the complex

$$R\Gamma_{\mathcal{H}}(K^\bullet) = \mathrm{Cone}^\bullet\big(\widehat{W}_0 K^\bullet_{\mathbb{Q}} \oplus F^0\widehat{W}_0 K^\bullet_{\mathbb{C}} \to \widehat{W}_0 K^\bullet_{\mathbb{C}}\big)[-1].$$

According to Kerr and Lewis [KL07, 2.8], the above cone complex can be seen as the total complex of the following double complex:

$$\mathcal{H}(r)^{i,j}_{U_p} := \begin{cases} 0, & j > 1 \\ \\ \ker(d) \subset \mathcal{D}^{2(r+1)}(Y_p^{[-i]}), & j = 1 \\ \\ \ker(d) \subset C^{2(r+i)}\big(Y_p^{[-i]}, \mathbb{Q}(r+i)\big) \bigoplus \\ \ker(d) \subset F^0\mathcal{D}^{2(r+1)}(Y_p^{[-i]}) \bigoplus \mathcal{D}^{2r+2i-i}(Y_p^{[-i]}), & j = 0 \\ \\ C_{\mathcal{D}}^{2r+2i+j}\big(Y_p^{[-i]}, \mathbb{Q}(r+i)\big), & j < 0 \end{cases}$$

where $Y_p^{[-i]}$'s are the components given by the semi-simplicial hyperresolution $Y_p^{[\bullet]} \to Y_p \to X_p$, associated to the normal crossing divisor Y_p and augmented over X_p. By [KL07, (2.10)], this construction defines a distinguished triangle

$$\mathbf{s}^\bullet\mathcal{H}_{Y_p}(r) \to \mathbf{s}^\bullet\mathcal{H}_{X_p}(r) \to \mathbf{s}^\bullet\mathcal{H}_{U_p}(r) \xrightarrow{+1}$$

where $\mathbf{s}^\bullet\mathcal{H}^{\bullet;\bullet}_{U_p}(r)$ computes the absolute Hodge cohomology of U_p. This construction defines for a semi-simplicial hyperresolution $U_\bullet \to U$ a triple complex, denote this complex by $\mathcal{H}_{U_\bullet}(r)$. First consider the double complex $\mathcal{H}^{i,j}_{U_p}(r)$ over U_p, then considering the total complex $\mathbf{s}^\bullet\mathcal{H}_{U_\bullet}(r)$ on each smooth component on the simplicial variety, we have the complex $\mathbf{s}^\bullet\mathcal{H}_U(r)$. The absolute Hodge cohomology of U is defined by any simplicial mixed Hodge complex of $U_\bullet$ via the descent given by the augmentation $a\colon U_\bullet \to U$.

− Using the extension criterion of Guillén-Navarro [GNA02], we can extend the definition of absolute Hodge cohomology from smooth varieties to complete singular varieties, or more generally, to quasi-projective varieties, using the hyperresolution of the pair $U \hookrightarrow X$.

Definition 4.2.10 For a quasi-projective variety U over $\mathbb{C}$, the **absolute Hodge cohomology** of U is defined by

$$H^q_{\mathcal{H}}\big(U, \mathbb{Q}(r)\big) := \mathbb{H}^q(U, \mathbf{s}Ra_*\mathcal{H}_{U_\bullet}(r)) = \mathbb{H}^q(U_\bullet, \mathcal{H}_{U_\bullet}(r)).$$

This definition is similar to one given in [Nav08, A.6], and is well-defined, because it is independent of the choice of the hyperresolution $U_\bullet \to U$. Moreover, is indepedent of the compactification $U_\bullet \hookrightarrow X_\bullet$ of the hyperresolution [Jan88]. There is a spectral sequence of absolute Hodge cohomology [BZ90]:

$$E_1^{p,q}(r) := H^q_{\mathcal{H}}(U_p, \mathbb{Q}(r)) \Rightarrow H^{p-q}_{\mathcal{H}}(U_\bullet, \mathbb{Q}(r)) = H^{p-q}_{\mathcal{H}}(U, \mathbb{Q}(r)).$$

The cohomological properties of smooth varieties naturally extend to the case of singular varieties, in addition the descent gives us a long exact sequence

$$\to H^{r-1}_{\mathcal{H}}(E,\mathbb{Q}(m)) \to H^r_{\mathcal{H}}(X,\mathbb{Q}(m)) \to H^r_{\mathcal{H}}(\Sigma,\mathbb{Q}(m)) \oplus H^r_{\mathcal{H}}(\widetilde{X},\mathbb{Q}(m)) \to$$

for a 2-resolution of the form

$$\begin{array}{ccc} E & \longrightarrow & \widetilde{X} \\ \downarrow & & \downarrow \\ \Sigma & \longrightarrow & X. \end{array}$$

4.3. A regulator for singular varieties

Let X be a projective, singular variety over $\mathbb{C}$. In this case, the construction of regulator is based on a diagram given by the cubical hyperresolution

$$\cdots \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} X_3 \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} X_2 \substack{\longrightarrow\\ \longrightarrow} X_1 \xrightarrow{a} X$$

which exists by resolution of singularities [GNPP88]. The basis of the construction is the model given by the case of a NCD [KL07, Pro. 8.12], with each X_p a smooth and projective variety. In this case, the absolute Hodge cohomology of X_p is computed by the Deligne complex. Then, the regulator morphism on the level of complexes is given by [KLM06, 5.5]:

$$\begin{array}{rcl} \mathcal{Z}^{p,q}(r) := \mathcal{Z}^{p,q}_{\mathbb{R}}(X_p,-q) & \xrightarrow{\text{KLM}} & \mathcal{H}^{p,q}(r) \\ \alpha & \mapsto & (-2\pi i)^{r+q}\big((2\pi i)^{-q}T_\alpha,\Omega_\alpha,R_\alpha\big). \end{array}$$

where the complex $\mathcal{H}^{p,q}(r)$ is quasi-isomorphic to Deligne complex. Organizing these morphisms in a double complex, it defines a fourth quadrant double complex. Since all components in the simplicial hyperresolution $X_\bullet \to X$ is smooth and projective, there is a filtration on the level of complexes given by the "weight" filtration on the absolute Hodge complex that induces a spectral sequence. Then, the regulator morphism and the Bloch's cycle-class morphism extend to cycle-class morphism of singular and projective varieties, defined by Hanamura's Chow cohomology groups.

Theorem 4.3.1 *There is a morphism of (double) complexes in the derived category $\mathcal{Z}^{p,q}(r) \to \mathcal{H}^{p,q}(r)$ given by the KLM-formula, with a morphism of* *total complexes* $\mathbb{Z}^{SF}_{X_\bullet}(r)[2r] \simeq \mathcal{Z}^r(X_\bullet)^* \to \mathcal{H}_{X_\bullet}(r)$*. This regulator morphism induces a cycle-class morphism on the total cohomologies*

$$H^{2r-*}_{\mathcal{M}}(X, \mathbb{Q}(r)) \cong \mathrm{CHC}^r(X,*) \to H^{2r-*}_{\mathcal{H}}(X, \mathbb{Q}(r)).$$

Such a morphism coincides, when X *is smooth, with the KLM-regulator [KLM06].*

Proof. Let X be a singular, projective variety over $\mathbb{C}$. Consider a cubical hyperresolution $X_\bullet \to X$, where each component X_p is smooth and projective. Under the quasi-isomorphism $\mathbb{Z}^{SF}(r)[2r](X_p) \simeq \mathcal{Z}^r(X_p \times \mathbb{A}^r, \bullet)$, and using the KLM-morphism we have a morphism of spectral sequences:

$$\begin{tikzcd}
H^{2r+q}_{\mathcal{M}}(X_p, \mathbb{Q}(r)) \arrow[r, Rightarrow] \arrow[d, "\cong"'] & H^{2r+q}_{\mathcal{M}}(X, \mathbb{Q}(r)) \arrow[d, "\cong"] \\
\mathrm{CH}^r(X_p, -q) \arrow[r, Rightarrow] \arrow[d, "\mathrm{cl}"'] & \mathrm{CHC}^r(X, p-q) \arrow[d, "\mathrm{cl}"] \\
H^{2r+q}_{\mathcal{H}}(X_p, \mathbb{Q}(r)) \arrow[r, Rightarrow] & H^{2r-p+q}_{\mathcal{H}}(X, \mathbb{Q}(r))
\end{tikzcd}$$

These are morphisms of E_1-pages of the spectral sequences induced by KLM-formula given in the E_0-page by $\mathcal{Z}^{p,q}_{\mathbb{R}}(X_p, -q) \to \mathcal{H}^{p,q}(r)$.

Example 4.3.2 For a projective curve C over $\mathbb{C}$, consider the normalization $\eta\colon \widetilde{C} \to C$ with singular locus $\Sigma \subset C$, and $E = \eta^{-1}(\Sigma)$ (with reduced scheme structure). We have a cartesian square

$$\begin{tikzcd}
E \arrow[r] \arrow[d] & \widetilde{C} \arrow[d, "\eta"] \\
\Sigma \arrow[r, "i"] & C
\end{tikzcd}$$

where i is the embedding. The total (cone) complex

$$\mathcal{Z}^r(C,\bullet)^* := \mathrm{Cone}\big\{\mathcal{Z}^r(\widetilde{C},\bullet) \oplus \mathcal{Z}^r(\Sigma,\bullet) \to \mathcal{Z}^r(E,\bullet)\big\}[-1]$$

computes the motivic cohomology of C. The absolute Hodge cohomology of C is given by the complex

$$\mathcal{H}^\bullet_C(r) := \mathrm{Cone}\big\{\mathcal{H}^{\bullet,\bullet}_{\widetilde{C}}(r) \oplus \mathcal{H}^{\bullet,\bullet}_{\Sigma}(r) \to \mathcal{H}^{\bullet,\bullet}_{E}(r)\big\}[-1].$$

By definition of the triangulated estructure, and realizations given on the level of double complexes [KL07, Ex. 3.1], we have the following regulator:

$$\mathcal{Z}^r(C,\bullet)^* \to \mathcal{H}^\bullet_C(r).$$

The above square induces a long exact sequence on absolute Hodge cohomology and motivic cohomology, by descent:

$$\begin{array}{ccccc}
\mathrm{CH}^r(E,m+1) & \longrightarrow & H^{2r-m}_{\mathcal{M}}(X,\mathbb{Q}(r)) & \longrightarrow & \mathrm{CH}^r(\widetilde{X},m)\oplus \mathrm{CH}^r(\Sigma,m) \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
H^{2r-m-1}_{\mathcal{H}}(E,\mathbb{Q}(r)) & \longrightarrow & H^{2r-m}_{\mathcal{H}}(X,\mathbb{Q}(r)) & \longrightarrow & H^{2r-m}_{\mathcal{H}}(\widetilde{X},\mathbb{Q}(r))\oplus H^{2r-m}_{\mathcal{H}}(\Sigma,\mathbb{Q}(r)).
\end{array}$$

Chapter 5

Some computations of motivic cohomology

Let X be a singular, quasi-projective variety. A cubical hyperresolution $X_\bullet \to X$ in the sense of [GNPP88] is a semi-simplicial scheme consisting of smooth schemes X_p for $0 \leq p \leq N$ with some N, and face morphisms $d_i \colon X_p \to X_{p-1}$ satisfying certain identities. The cubical hyperresolution induces a fourth quadrant double complex

$$\mathcal{Z}^r(X_0, \bullet) \xrightarrow{d^*} \mathcal{Z}^r(X_1, \bullet) \xrightarrow{d^*} \cdots \xrightarrow{d^*} \mathcal{Z}^r(X_N, \bullet).$$

The cohomological cycle complex $\mathcal{Z}^r(X_\bullet)^*$ of X is the total complex of above double complex, and the Chow cohomology group (or motivic cohomology) is by definition the m^{th}-homology of this complex: $\mathrm{CHC}^r(X, m) = H_m\big(\mathcal{Z}^r(X, \bullet)^*\big)$. There is an associated spectral sequence

$$E_1^{p,q}(r) := \mathrm{CH}^r(X_p, -q) \Rightarrow \mathrm{CHC}^r(X, p-q) \cong H_{\mathcal{M}}^{2r-p+q}(X, \mathbb{Z}(r))$$

which does not depend of the choice of cubical hyperresolution $X_\bullet \to X$ [Han00]. Let X be a normal variety of dimension $d \geq 3$. Recall that the singular locus X_{sing} is of codimension ≥ 2. The resolution of singularities of X along X_{sing} induces an exact sequence of motivic cohomology groups:

$$\to H_{\mathcal{M}}^{2r-m}(X, \mathbb{Z}(r)) \to \mathrm{CH}^r(\widetilde{X}, m) \oplus \mathrm{CH}^r(X_{\text{sing}}, m) \to \mathrm{CH}^r(E, m) \to$$

When X is a smooth, the Bloch's higher Chow groups vanish in negative degrees. In codimension $r = 1$, we have $H_{\mathcal{M}}^{2-m}(X, \mathbb{Z}(1)) = 0$ for $m \neq 0, 1$; $H_{\mathcal{M}}^{2}(X, \mathbb{Z}(1)) = \mathrm{Pic}(X)$ and $H_{\mathcal{M}}^{1}(X, \mathbb{Z}(1)) = \Gamma(X, \mathcal{O}_X^*)$ [Blo86a]. This fact implies, for any variety, that $\mathrm{CHC}^1(X, m) = 0$ for $m > 1$.

5.1. Motivic cohomology of varieties of dimension 3

In this section, consider an irreducible normal quasi-projective variety X of dimension 3 such that the singular locus X_{sing} is either a finite number of points or a curve. For varieties of lower dimensions, see [Han14].

Example 5.1.1 Let X be an irreducible projective variety of dimension 3 over an algebracally closed field k. If the singular locus X_{sing} is a connected smooth curve, consider the resolution of singularities along X_{sing}. This induces a Cartesian square

$$\begin{tikzcd} E \arrow[r] \arrow[d] & \widetilde{X} \arrow[d, "p"] \\ X_{\text{sing}} \arrow[r, "i"] & X \end{tikzcd}$$

and we suppose that $E = p^{-1}(X_{\text{sing}})$ is a connected smooth surface. Therefore, we have a cubical hyperresolution

$$X_2 = E \rightrightarrows X_1 = \widetilde{X} \coprod X_{\text{sing}} \xrightarrow{a} X$$

with cohomological cycle complex

$$\mathcal{Z}^r(X_\bullet)^* = \text{Cone}\big\{\mathcal{Z}^r(\widetilde{X}, \bullet) \oplus \mathcal{Z}^r(X_{\text{sing}}, \bullet) \to \mathcal{Z}^r(E, \bullet)\big\}[-1].$$

In codimension $r = 1$, we have a long exact sequence

$$0 \to \text{CHC}^1(X, 1) \to \Gamma(\widetilde{X}, \mathcal{O}^*_{\widetilde{X}}) \oplus \Gamma(X_{\text{sing}}, \mathcal{O}^*_{X_{\text{sing}}}) \to \Gamma(E, \mathcal{O}^*_E) \to$$

$$\text{CHC}^1(X) \to \text{CH}^1(\widetilde{X}) \oplus \text{CH}^1(X_{\text{sing}}) \to \text{CH}^1(E) \to \text{CHC}^1(X, -1) \to 0.$$

Under the identification (2.2.4), the above sequence gives us the following:

$$0 \to \text{CHC}^1(X, 1) \to k^* \oplus k^* \to k^* \to \text{CHC}^1(X) \to$$

$$\to \text{Pic}(\widetilde{X}) \oplus \text{Pic}(X_{\text{sing}}) \to \text{Pic}(E) \to \text{CHC}^1(X, -1) \to 0.$$

Then $\text{CHC}^1(X, m) = 0$ for $m \neq -1, 0, 1$, and $\text{CHC}^1(X, 1) = k^*$. It is also immediate to see that

$$\text{CHC}^1(X) = \text{Ker}\{\text{Pic}(\widetilde{X}) \oplus \text{Pic}(X_{\text{sing}}) \to \text{Pic}(E)\}.$$

By the same argument we have

$$\text{CHC}^1(X, -1) = \text{Coker}\{\text{Pic}(\widetilde{X}) \oplus \text{Pic}(X_{\text{sing}}) \to \text{Pic}(E)\}.$$

In particular, if $\mathcal{E}$ is a vector bundle of rank $n + 1$ over X_{sing} with $E = \mathbb{P}(\mathcal{E}) \to X_{\text{sing}}$ its projectivization, then

$$\text{Pic}(E) \cong \text{Pic}(X_{\text{sing}}) \oplus \mathbb{Z}.$$

In this case, the group $\text{CHC}^1(X)$ can be seen as:

$$\text{CHC}^1(X) \cong \text{Ker}\{\text{Pic}(\widetilde{X}) \to \mathbb{Z}\}$$

and $\text{CHC}^1(X, -1)$ is a torsion group.

Example 5.1.2 Let X be a singular projective variety of dimension 3, with singular locus $X_{\text{sing}} = \{p\}$. Let $p\colon \widetilde{X} \to X$ be a desingularization of X such that $E = p^{-1}(X_{\text{sing}})$ is a normal crossing divisor. Then we have a Cartesian square

$$\begin{tikzcd} E \arrow[r] \arrow[d] & \widetilde{X} \arrow[d, "p"] \\ X_{\text{sing}} \arrow[r] & X \end{tikzcd}$$

this is a 2-resolution of X. This desingularization extends to a cubical hyperresolution by resolving $E \to X_{\text{sing}}$, but in this case X_{sing} is non-singular and the process continues via a 2-resolution of E. This process reduces the study of motivic cohomology and higher Chow groups to E. By definition, $\mathrm{CHC}^r(X, m)$ is the m^{th} homology of the complex

$$\mathcal{Z}^r(X_\bullet, \bullet)^* := \mathrm{Cone}\{\mathcal{Z}^r(\widetilde{X}, \bullet) \oplus \mathcal{Z}^r(X_{\text{sing}}, \bullet) \to \mathcal{Z}^r(E, \bullet)^*\}[-1].$$

Then, we have a long exact sequence

$$\begin{aligned}
\cdots &\longrightarrow \mathrm{CH}^1(\widetilde{X}, 3) \oplus \mathrm{CH}^1(X_{\text{sing}}, 3) \longrightarrow \mathrm{CHC}^1(E, 3) \longrightarrow \mathrm{CHC}^1(X, 2) \\
&\longrightarrow \mathrm{CH}^1(\widetilde{X}, 2) \oplus \mathrm{CH}^1(X_{\text{sing}}, 2) \longrightarrow \mathrm{CHC}^1(E, 2) \longrightarrow \mathrm{CHC}^1(X, 1) \\
&\longrightarrow \mathrm{CH}^1(\widetilde{X}, 1) \oplus \mathrm{CH}^1(X_{\text{sing}}, 1) \longrightarrow \mathrm{CHC}^1(E, 1) \longrightarrow \mathrm{CHC}^1(X) \\
&\longrightarrow \mathrm{CH}^1(\widetilde{X}) \oplus \mathrm{CH}^1(X_{\text{sing}}) \longrightarrow \mathrm{CHC}^1(E) \longrightarrow \mathrm{CHC}^1(X, -1) \\
&\longrightarrow \mathrm{CH}^1(\widetilde{X}, -1) \oplus \mathrm{CH}^1(X_{\text{sing}}, -1) \to \mathrm{CHC}^1(E, -1) \to \mathrm{CHC}^1(X, -2) \to \cdots
\end{aligned} \tag{5.1}$$

The first observation is that $\mathrm{CHC}^1(E, m) \cong \mathrm{CHC}^1(X, m-1)$ for $m \neq 0, 1, 2$. To compute $\mathcal{Z}^r(E, \bullet)^*$ consider the following. A strict normal crossing divisor $E = \bigcup_{i=1}^N E_i$, with each component an irreducible and projective surface. Further suppose that E is a tree, i.e. the associated graph is a tree. Let $E_{ij} = E_i \cap E_j$ be the intersection of any two irreducible components of E, so the intersections E_{ij} are curves. Let $E_{[1]} := \coprod E_i$ and $E_{[2]} := \coprod_{i<j} E_{ij}$. In our case, this description defines a simplicial scheme over E:

$$\coprod E_i \cap E_j \rightrightarrows \coprod E_i \xrightarrow{a} E.$$

The associated cohomological cycle complex is

$$\mathcal{Z}^r(E, \bullet)^* := \mathrm{Cone}^\bullet\big\{\mathcal{Z}^r(E_{[1]}, \bullet) \xrightarrow{\phi} \mathcal{Z}^r(E_{[2]}, \bullet)\big\}[-1].$$

Since $E_{[0]}$ and $E_{[1]}$ are projective and smooth, we have a long exact sequence of the form:

$$0 \to \mathrm{CHC}^1(E,1) \to \mathrm{CH}^1(E_{[0]},1) \xrightarrow{\phi^*} \mathrm{CH}^1(E_{[1]},1) \to \\ \mathrm{CHC}^1(E) \to \mathrm{CH}^1(E_{[0]}) \xrightarrow{\phi^*} \mathrm{CH}^1(E_{[1]}) \to \mathrm{CHC}^1(E,-1) \to 0. \quad (5.2)$$

The *associated graph* Γ of E is the graph consisting of the vertices corresponding to the components E_i, and the edges corresponding to intersections $E_{ij} = E_i \cap E_j$. The resulting graph Γ is connected since E connected. The cochain complex $C^\bullet(\Gamma)$ is the complex of two terms $d\colon C^0(\Gamma) \to C^1(\Gamma)$, where $C^i(\Gamma) = \mathrm{Hom}(C_i(\Gamma), \mathbb{Z})$ with $i = 0, 1$, is free abelian group with dual basis to $C_0(\Gamma)$ and $C_1(\Gamma)$ (generated by $\{E_i\}$ and $\{E_{ij}\}$ respectively), and $d(e_i) = \sum_{i<j} e_{ij} - \sum_{m<i} e_{mi}$. By definition $H^*(\Gamma)$ is the cohomology of the complex $C^\bullet(\Gamma)$ [Han14]. With this description, we have the following result:

Proposition 5.1.3 *Let E be a connected simple normal crossing divisor such that the associated graph is a tree, and $H^1(\Gamma) = 0$. Then, $\mathrm{CHC}^1(E, m) = 0$ for $m \neq -1, 0, 1$, and $\mathrm{CHC}^1(E, 1) = k^*$.*

Proof. Under the identification $\mathrm{CH}^1(E_{[0]}) = C^0(\Gamma) \otimes k^*$ and $\mathrm{CH}^1(E_{[1]}) = C^1(\Gamma) \otimes k^*$, the previous exact sequence (5.2) is transformed into the following:

$$0 \to \mathrm{CHC}^1(E,1) \to C^0(\Gamma) \otimes k^* \xrightarrow{\phi^*} C^1(\Gamma) \otimes k^*$$

$$\to \mathrm{CHC}^1(E) \to \mathrm{Pic}(E_{[0]}) \xrightarrow{\phi^*} \mathrm{Pic}(E_{[1]}) \to \mathrm{CHC}^1(E,-1) \to 0.$$

Since $H^1(\Gamma) = 0$, the morphism $\phi^*\colon C^0(\Gamma) \otimes k^* \to C^1(\Gamma) \otimes k^*$ is surjective

Corollary 5.1.4 *Let X be an irreducible normal projective variety of dimension 3 over an algebraically closed field k of characteristic zero. Suppose further that under the resolution of singularities, E is as in the above proposition. Then $\mathrm{CHC}^1(X, m) = 0$ for $m \neq -2, -1, 0, 1$, $\mathrm{CHC}^1(X, 1) = k^*$, and there is an exact sequence*

$$0 \to \mathrm{CHC}^1(X,1) \to k^* \oplus k^* \to k^* \to \mathrm{CHC}^1(X) \to \mathrm{Pic}(\widetilde{X}) \to$$

$$\mathrm{CHC}^1(E) \to \mathrm{CHC}^1(X,-1) \to 0 \to \mathrm{CHC}^1(E,-1) \to \mathrm{CHC}^1(X,-2) \to 0$$

Proof. By (5.1.3) and previous results, the long exact sequence (5.1) can be reduced to the following exact sequence:

$$0 \to \mathrm{CHC}^1(X,1) \to \mathrm{CH}^1(\widetilde{X},1) \oplus \mathrm{CH}^1(X_{\mathrm{sing}},1) \to \mathrm{CHC}^1(E,1) \to$$
$$\to \mathrm{CHC}^1(X) \to \mathrm{CH}^1(\widetilde{X}) \to \mathrm{CHC}^1(E) \to \mathrm{CHC}^1(X,-1) \to$$
$$\to \mathrm{CH}^1(\widetilde{X},-1) \oplus \mathrm{CH}^1(X_{\mathrm{sing}},-1) \to \mathrm{CHC}^1(E,-1) \to \mathrm{CHC}^1(X,-2) \to 0$$

Since the higher Chow groups of smooth varieties vanish in negative degrees, and $\mathrm{Pic}(X_{\mathrm{sing}}) = 0$, the corollary follows immediately. □

5.2. Varieties of higher dimension

Let X be a complex projective variety of dimension d, with singular locus X_{sing} smooth, irreducible of $\mathrm{codim}(X_{\mathrm{sing}}) \geq 2$. Consider a resolution of singularities $p\colon \widetilde{X} \to X$, where $E = p^{-1}(X_{\mathrm{sing}})$ is a simple normal crossing divisor such that the graph associated is a tree. This defines a 2-resolution $E \rightrightarrows \widetilde{X} \coprod X_{\mathrm{sing}} \to X$. Again, the cohomological cycle complex is given by

$$\mathcal{Z}^r(X_\bullet,\bullet)^* := \mathrm{Cone}\{\mathcal{Z}^r(\widetilde{X},\bullet) \oplus \mathcal{Z}^r(X_{\mathrm{sing}},\bullet) \to \mathcal{Z}^r(E,\bullet)^*\}[-1].$$

And there exists an exact sequence

$$\to \mathrm{CHC}^1(X,m) \to \mathrm{CHC}^1(\widetilde{X},m) \oplus \mathrm{CHC}^1(X_{\mathrm{sing}},m) \to \mathrm{CHC}^1(E,m) \to$$

Then the analysis is reduced to E. Since the associated graph to E is a tree with $H^1(\Gamma) = 0$, then $\mathrm{CHC}^1(E,m) = 0$ for $m \neq -1,0,1$, and $\mathrm{CHC}^1(E,1) = k^*$. It immediately follows that $\mathrm{CHC}^1(X,1) = k^*$.

Proposition 5.2.1 *There is an exact sequence:*

$$0 \to \mathrm{CHC}^1(X,1) \to k^* \oplus k^* \to k^* \to \mathrm{CHC}^1(X) \to \mathrm{Pic}(\widetilde{X}) \oplus \mathrm{Pic}(X_{\mathrm{sing}})$$
$$\to \mathrm{CHC}^1(E) \to \mathrm{CHC}^1(X,-1) \to 0 \to \mathrm{CHC}^1(E,-1) \to \mathrm{CHC}^1(X,-2) \to 0$$

with $\mathrm{CHC}^1(X,m) = 0$ *if* $m \neq -2,-1,0,1$, *and* $\mathrm{CHC}^1(X,1) = k^*$.

Index

Bibliography

[And04] Yves André, *Une introduction aux motifs (motifs purs, motifs mixtes, périodes)*, Panoramas et Synthèses [Panoramas and Syntheses], vol. 17, Société Mathématique de France, Paris, 2004. MR2115000 65

[Bei84] A. A. Beĭlinson, *Higher regulators and values of L-functions*, Current problems in mathematics, Vol. 24, Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1984, pp. 181–238. MR760999 xii, xiii

[Bei86] ———, *Notes on absolute Hodge cohomology*, Applications of algebraic K-theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983), Contemp. Math., vol. 55, Amer. Math. Soc., Providence, RI, 1986, pp. 35–68. MR862628 https://doi.org/10.1090/conm/055.1/862628 xiii, xiv, 30, 31

[Bei87] ———, *Height pairing between algebraic cycles*, K-theory, arithmetic and geometry (Moscow, 1984–1986), Lecture Notes in Math., vol. 1289, Springer, Berlin, 1987, pp. 1–25. MR923131 https://doi.org/10.1007/BFb0078364 xi, 60, 61

[Blo86a] Spencer Bloch, *Algebraic cycles and higher K-theory*, Adv. in Math. **61** (1986), no. 3, 267–304. MR852815 https://doi.org/10.1016/0001-8708(86)90081-2 xii, 24, 25, 27, 28, 79

[Blo86b] ———, *Algebraic cycles and the Beĭlinson conjectures*, The Lefschetz centennial conference, Part I (Mexico City, 1984), Contemp. Math., vol. 58, Amer. Math. Soc., Providence, RI, 1986, pp. 65–79. MR860404 https://doi.org/10.1090/conm/058.1/860404 xiv, 25, 35

[Bon09] M. V. Bondarko, *Differential graded motives: weight complex, weight filtrations and spectral sequences for realizations; Voevodsky versus Hanamura*, J. Inst. Math. Jussieu **8** (2009), no. 1, 39–97. MR2461902
https://doi.org/10.1017/S147474800800011X 66

[BS01] Paul Balmer and Marco Schlichting, *Idempotent completion of triangulated categories*, J. Algebra **236** (2001), no. 2, 819–834. MR1813503
https://doi.org/10.1006/jabr.2000.8529 62

[BZ90] Jean-Luc Brylinski and Steven Zucker, *An overview of recent advances in Hodge theory*, Several complex variables, VI, Encyclopaedia Math. Sci., vol. 69, Springer, Berlin, 1990, pp. 39–142. MR1095090 74

[Car80] James A. Carlson, *Extensions of mixed Hodge structures*, Journées de Géometrie Algébrique d'Angers, Juillet 1979/Algebraic Geometry, Angers, 1979, Sijthoff & Noordhoff, Alphen aan den Rijn—Germantown, Md., 1980, pp. 107–127. MR605338 15, 23, 24

[Dur83] Alan H. Durfee, *A naive guide to mixed Hodge theory*, Singularities, Part 1 (Arcata, Calif., 1981), Proc. Sympos. Pure Math., vol. 40, Amer. Math. Soc., Providence, RI, 1983, pp. 313–320. MR713069 12

[EV88] Hélène Esnault and Eckart Viehweg, *Deligne-Beĭlinson cohomology*, Beĭlinson's conjectures on special values of L-functions, Perspect. Math., vol. 4, Academic Press, Boston, MA, 1988, pp. 43–91. MR944991 17, 20

[FS02] Eric M. Friedlander and Andrei Suslin, *The spectral sequence relating algebraic K-theory to motivic cohomology*, Ann. Sci. École Norm. Sup. (4) **35** (2002), no. 6, 773–875. MR1949356
https://doi.org/10.1016/S0012-9593(02)01109-6 56

[Ful84] William Fulton, *Intersection theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 2, Springer-Verlag, Berlin, 1984. MR732620
https://doi.org/10.1007/978-3-662-02421-8 25

[FV00] Eric M. Friedlander and Vladimir Voevodsky, *Bivariant cycle cohomology*, Cycles, transfers, and motivic homology theories, Ann. of Math. Stud., vol. 143, Princeton Univ. Press, Princeton, NJ, 2000, pp. 138–187. MR1764201 xii, xvi, 27, 28, 56, 57, 58, 59

[GGK10] Mark Green, Phillip Griffiths, and Matt Kerr, *Néron models and limits of Abel-Jacobi mappings*, Compos. Math. **146** (2010), no. 2, 288–366. MR2601630 https://doi.org/10.1112/S0010437X09004400 68

[GH78] Phillip Griffiths and Joseph Harris, *Principles of algebraic geometry*, Wiley-Interscience [John Wiley & Sons], New York, 1978, Pure and Applied Mathematics. MR507725 4, 5

[GNA02] Francisco Guillén and Vicente Navarro Aznar, *Un critère d'extension des foncteurs définis sur les schémas lisses*, Publ. Math. Inst. Hautes Études Sci. (2002), no. 95, 1–91. MR1953190 https://doi.org/10.1007/s102400200003 xv, 43, 55, 67, 71, 72, 74

[GNPP88] F. Guillén, V. Navarro Aznar, P. Pascual Gainza, and F. Puerta, *Hyperrésolutions cubiques et descente cohomologique*, Lecture Notes in Mathematics, vol. 1335, Springer-Verlag, Berlin, 1988, Papers from the Seminar on Hodge-Deligne Theory held in Barcelona, 1982. MR972983 https://doi.org/10.1007/BFb0085054 xiv, xv, 12, 20, 28, 43, 45, 46, 47, 50, 67, 69, 72, 75, 79

[Gon02] A. B. Goncharov, *Explicit regulator maps on polylogarithmic motivic complexes*, Motives, polylogarithms and Hodge theory, Part I (Irvine, CA, 1998), Int. Press Lect. Ser., vol. 3, Int. Press, Somerville, MA, 2002, pp. 245–276. MR1978709 xiv

[Gon95] ———, *Chow polylogarithms and regulators*, Math. Res. Lett. **2** (1995), no. 1, 95–112. MR1312980 https://doi.org/10.4310/MRL.1995.v2.n1.a9 xiv, 36

[Han00] Masaki Hanamura, *Homological and cohomological motives of algebraic varieties*, Invent. Math. **142** (2000), no. 2, 319–349. MR1794065 https://doi.org/10.1007/s002220000091 xii, xv, 28, 43, 51, 56, 58, 61, 72, 79

[Han14] ———, *Chow cohomology groups of algebraic surfaces*, Math. Res. Lett. **21** (2014), no. 3, 479–493. MR3272025 https://doi.org/10.4310/MRL.2014.v21.n3.a6 xvi, 55, 79, 82

[HanI] ———, *Mixed motives and algebraic cycles. I*, Math. Res. Lett. **2** (1995), no. 6, 811–821. MR1362972 https://doi.org/10.4310/MRL.1995.v2.n6.a12 60, 66

[HanII] ———, *Mixed motives and algebraic cycles. II*, Invent. Math. **158** (2004), no. 1, 105–179. MR2090362 https://doi.org/10.1007/s00222-004-0369-0 60

[HanIII] ———, *Mixed motives and algebraic cycles. III*, Math. Res. Lett. **6** (1999), no. 1, 61–82. MR1682709 https://doi.org/10.4310/MRL.1999.v6.n1.a5 60

[Hir64] Heisuke Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II*, Ann. of Math. (2) **79** (1964), 109–203; ibid. (2) **79** (1964), 205–326. MR0199184 https://doi.org/10.2307/1970547 19, 43, 47, 71

[HodgeI] Pierre Deligne, *Théorie de Hodge. I*, Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1, 1971, pp. 425–430. MR0441965 xi, 9

[HodgeII] ———, *Théorie de Hodge. II*, Inst. Hautes Études Sci. Publ. Math. (1971), no. 40, 5–57. MR0498551 http://www.numdam.org/item?id=PMIHES_1971__40__5_0 xi, 4, 8, 9, 11, 12, 39

[HodgeIII] ———, *Théorie de Hodge. III*, Inst. Hautes Études Sci. Publ. Math. (1974), no. 44, 5–77. MR0498552 http://www.numdam.org/item?id=PMIHES_1974__44__5_0 xi, xiii, 9, 11, 13, 30, 39, 43, 69, 70, 71, 72

[Hub00] Annette Huber, *Realization of Voevodsky's motives*, J. Algebraic Geom. **9** (2000), no. 4, 755–799. MR1775312 32

[Jan88] Uwe Jannsen, *Deligne homology, Hodge-$\mathcal{D}$-conjecture, and motives*, Beĭlinson's conjectures on special values of L-functions, Perspect. Math., vol. 4, Academic Press, Boston, MA, 1988, pp. 305–372. MR944998 17, 20, 21, 31, 32, 74

[KL07] Matt Kerr and James D. Lewis, *The Abel-Jacobi map for higher Chow groups. II*, Invent. Math. **170** (2007), no. 2, 355–420. MR2342640
https://doi.org/10.1007/s00222-007-0066-x xiv, 23, 28, 29, 30, 32, 37, 41, 42, 67, 68, 71, 73, 74, 75, 77

[KLL18] Matt Kerr, James Lewis, and Patrick Lopatto, *Simplicial Abel-Jacobi maps and reciprocity laws*, J. Algebraic Geom. **27** (2018), no. 1, 121–172, With an appendix by José Ignacio Burgos-Gil. MR3722692
https://doi.org/10.1090/jag/692 36

[KLM06] Matt Kerr, James D. Lewis, and Stefan Müller-Stach, *The Abel-Jacobi map for higher Chow groups*, Compos. Math. **142** (2006), no. 2, 374–396. MR2218900
https://doi.org/10.1112/S0010437X05001867 xiv, 23, 28, 32, 34, 35, 36, 37, 67, 68, 75, 76

[Lev94] Marc Levine, *Bloch's higher Chow groups revisited*, Astérisque (1994), no. 226, 10, 235–320, K-theory (Strasbourg, 1992). MR1317122 25, 27, 66

[Lev98] ———, *Mixed motives*, Mathematical Surveys and Monographs, vol. 57, American Mathematical Society, Providence, RI, 1998. MR1623774
https://doi.org/10.1090/surv/057 28, 32, 43, 52, 60, 61

[Lew16] James D. Lewis, *Lectures on Hodge theory and algebraic cycles*, Commun. Math. Stat. **4** (2016), no. 2, 93–188. MR3513354
https://doi.org/10.1007/s40304-015-0080-z 29

[Lic84] S. Lichtenbaum, *Values of zeta-functions at nonnegative integers*, Number theory, Noordwijkerhout 1983 (Noordwijkerhout, 1983), Lecture Notes in Math., vol. 1068, Springer, Berlin, 1984, pp. 127–138. MR756089
https://doi.org/10.1007/BFb0099447 xi, 61

[MNP13] Jacob P. Murre, Jan Nagel, and Chris A. M. Peters, *Lectures on the theory of pure motives*, University Lecture Series, vol. 61, American Mathematical Society, Providence, RI, 2013. MR3052734
https://doi.org/10.1090/ulect/061 64

[MVW06] Carlo Mazza, Vladimir Voevodsky, and Charles Weibel, *Lecture notes on motivic cohomology*, Clay Mathematics Monographs, vol. 2, American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2006. MR2242284 xii, xiii, xvi, 28, 56, 57, 59, 60, 61, 63, 65

[Nag62] Masayoshi Nagata, *Imbedding of an abstract variety in a complete variety*, J. Math. Kyoto Univ. **2** (1962), 1–10. MR142549 https://doi.org/10.1215/kjm/1250524969 19, 71

[Nav08] A. Navarro, *Riemann-Roch for homotopy invariant K-theory and Gysin morphisms*, Adv. Math. **328** (2018), 501–554. MR3771136 https://doi.org/10.1016/j.aim.2018.01.001 74

[NS89] Yu. P. Nesterenko and A. A. Suslin, *Homology of the general linear group over a local ring, and Milnor's K-theory*, Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), no. 1, 121–146. MR992981 https://doi.org/10.1070/IM1990v034n01ABEH000610 27

[PS08] Chris A. M. Peters and Joseph H. M. Steenbrink, *Mixed Hodge structures*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 52, Springer-Verlag, Berlin, 2008. MR2393625 9, 10, 14, 15, 43, 47

[Tot92] Burt Totaro, *Milnor K-theory is the simplest part of algebraic K-theory*, K-Theory **6** (1992), no. 2, 177–189. MR1187705 https://doi.org/10.1007/BF01771011 27

[Voe00] Vladimir Voevodsky, *Triangulated categories of motives over a field*, Cycles, transfers, and motivic homology theories, Ann. of Math. Stud., vol. 143, Princeton Univ. Press, Princeton, NJ, 2000, pp. 188–238. MR1764202 27, 60, 65, 66

[VSF00] Vladimir Voevodsky, Andrei Suslin, and Eric M. Friedlander, *Cycles, transfers, and motivic homology theories*, Annals of Mathematics Studies, vol. 143, Princeton University Press, Princeton, NJ, 2000. MR1764197 56, 61